

Ordinary-derivative formulation of conformal low spin fields

R.R. Metsaev*

*Department of Theoretical Physics, P.N. Lebedev Physical Institute,
Leninsky prospect 53, Moscow 119991, Russia*

Abstract

Conformal fields in flat space-time of even dimension greater than or equal to four are studied. Second-derivative formulation for spin 0, 1, 2 conformal bosonic fields and first-derivative formulation for spin 1/2, 3/2 conformal fermionic fields are developed. For spin 1, 3/2, 2 conformal fields, we obtain gauge invariant Lagrangians and corresponding gauge transformations. Gauge symmetries are realized by involving the Stueckelberg fields. Realization of global conformal boost symmetries is obtained. Also, we demonstrate use of the light-cone gauge for counting of on-shell degrees of freedom in higher-derivative conformal field theories.

* E-mail: metsaev@lpi.ru

1 Introduction

In view of the aesthetic features of conformal field theory a interest in this theory was periodically renewed (see [1] and references therein). Conjectured duality [2] of large N conformal $\mathcal{N} = 4$ SYM theory and type IIB superstring theory in $AdS_5 \times S^5$ has triggered intensive and in-depth study of various aspects of conformal fields. Conformal fields in space-time of dimension $d \geq 4$ can be separated into two groups: fundamental conformal fields and shadow fields. This is to say that field having Lorentz algebra spin s and conformal dimension $\Delta \geq \Delta_0$, $\Delta_0 = s + d - 2$, is referred to as fundamental field¹, while field having Lorentz algebra spin s and dual conformal dimension $\hat{\Delta} = d - \Delta$ is sometimes referred to as shadow field. It is the shadow fields that are used to discuss conformally invariant equations of motion and Lagrangian formulations (see e.g. [1],[4],[5]²). In the framework of AdS/CFT correspondence, the shadow fields manifest themselves in two related ways at least. Firstly, they appear as boundary values of non-normalizable solution of equations of motion for bulk fields of IIB supergravity in $AdS_5 \times S^5$ background (see e.g. [7]-[11]³). Secondly, shadow fields appearing in spin 2 field supermultiplet of $N = 4$ superconformal algebra constitute multiplet of conformal supergravity. The conformal supergravity multiplet couples with fields of $\mathcal{N} = 4$ supersymmetric YM theory. It turns out that IIB supergravity expanded over AdS background and evaluated over Dirichlet problem reproduces of action of $N = 4$ conformal supergravity [14]⁴. Note also that $N = 4$ conformal supergravity share the same global supersymmetries, though realized in different way, as supergravity/superstring theory in $AdS_5 \times S^5$ Ramond-Ramond background. In view of these relations to IIB supergravity/superstring in $AdS_5 \times S^5$ and supersymmetric YM theory we think that various alternative formulations of shadow fields will be useful to understand string/gauge theory dualities better. In this paper we deal only with shadow fields. These fields will be referred to as conformal fields in what follows.

Lagrangian formulations of most conformal fields involve higher derivatives. The purpose of this paper is to develop ordinary-derivative, gauge invariant, and Lagrangian formulation for free conformal fields. This is to say that our Lagrangians for free bosonic fields do not involve higher than second order terms in derivatives, while our Lagrangians for free fermionic fields do not involve higher than first order terms in derivatives. As is well known arbitrary higher-derivative theory can be rewritten in ordinary-derivative form by introducing additional field degrees of freedom. Our purpose is to introduce the additional field degrees of freedom so

¹We note that fundamental conformal fields with $s = 1$, $\Delta = d - 1$ and $s = 2$, $\Delta = d$, correspond to conserved vector current and conserved traceless spin two tensor field (energy-momentum tensor) respectively. Conserved conformal currents can be built from massless scalar, spinor and spin 1 fields (see e.g. [3]).

² Discussion of equations for mixed symmetry conformal fields with discrete Δ may be found in [6].

³In earlier literature, discussion of shadow field dualities may be found in [12, 13].

⁴Also, conformal symmetries manifest themselves in the tensionless limit of strings [15] (see also [16, 17]).

that to respect the following three requirements: i) ordinary-derivative formulation should be Lagrangian; ii) the additional fields should be supplemented by appropriate additional gauge symmetries so that to retain on-shell D.o.F of the generic higher-derivative theory⁵; iii) realization of global conformal symmetries should be local.

In this paper, we discuss ordinary-derivative formulation of free low spin conformal fields in space-time of even dimension $d \geq 4$. Ordinary-derivative theory of interacting spin 2 conformal field in $4d$ general gravitational background, (i.e. ordinary-derivative form of the $4d$ conformal Weyl gravity) is discussed in Appendix C⁶.

2 Preliminaries

2.1 Notation

Our conventions are as follows. x^A denotes coordinates in d -dimensional flat space-time, while ∂_A denotes derivative with respect to x^A , $\partial_A \equiv \partial/\partial x^A$. Vector indices of the Lorentz algebra $so(d-1, 1)$ take the values $A, B, C, E = 0, 1, \dots, d-1$. We use $2^{[d/2]} \times 2^{[d/2]}$ Dirac gamma matrices γ^A in d -dimensions, $\{\gamma^A, \gamma^B\} = 2\eta^{AB}$, $\gamma^{A\dagger} = \gamma^0 \gamma^A \gamma^0$, where η^{AB} is mostly positive flat metric tensor. To simplify our expressions we drop η_{AB} in scalar products, i.e. we use $X^A Y^A \equiv \eta_{AB} X^A Y^B$. We adopt the notation $\square = \partial^A \partial_A$, $\not{\partial} \equiv \gamma^A \partial_A$, $\alpha \partial = \alpha^A \partial_A$, $\gamma \alpha = \gamma^A \alpha^A$, $\alpha^2 = \alpha^A \alpha^A$.

To avoid complicated tensor expressions we use a set of the creation operators $\alpha^A, \zeta, v^\oplus, v^\ominus$, and the respective set of annihilation operators $\bar{\alpha}^A, \bar{\zeta}, \bar{v}^\ominus, \bar{v}^\oplus$,

$$\bar{\alpha}^A|0\rangle = 0, \quad \bar{\zeta}|0\rangle = 0, \quad \bar{v}^\oplus|0\rangle = 0, \quad \bar{v}^\ominus|0\rangle = 0. \quad (2.1)$$

These operators satisfy the commutators

$$[\bar{\alpha}^A, \alpha^B] = \eta^{AB}, \quad [\bar{\zeta}, \zeta] = 1, \quad (2.2)$$

$$[\bar{v}^\oplus, v^\ominus] = 1, \quad [\bar{v}^\ominus, v^\oplus] = 1, \quad (2.3)$$

⁵ To realize those additional gauge symmetries we adopt the approach of Refs.[18, 19] which turns out to be the most useful for our purposes.

⁶ As a side of remark we note intriguing matching of spin 2 fields content of ordinary-derivative form of $4d$ conformal Weyl gravity and massless higher-spin AdS_4 field theory. This this to say that the ordinary-derivative form of conformal Weyl gravity involves, besides spin 1 field, one metric tensor field and one rank-2 symmetric tensor field. The same spin 2 fields content (i.e. one metric tensor field and one rank-2 symmetric tensor field) appears in massless higher-spin AdS_4 field theory [20]. So a natural question whether this matching of spin 2 fields content is accident. Recent interesting discussion of conformal symmetries of massless higher-spin AdS_4 field theory may be found in [21].

and will often be referred to as oscillators in what follows⁷. The oscillators α^A , $\bar{\alpha}^A$ and ζ , $\bar{\zeta}$, v^\oplus , v^\ominus , \bar{v}^\oplus , \bar{v}^\ominus transform in the respective vector and scalar representations of the $so(d-1, 1)$ Lorentz algebra and satisfy the following hermitian conjugation rules:

$$\alpha^{A\dagger} = \bar{\alpha}^A, \quad \zeta^\dagger = \bar{\zeta}, \quad (2.4)$$

$$v^{\oplus\dagger} = \bar{v}^\oplus, \quad v^{\ominus\dagger} = \bar{v}^\ominus. \quad (2.5)$$

Throughout this paper we use operators constructed out the oscillators,

$$N_\alpha \equiv \alpha^A \bar{\alpha}^A, \quad (2.6)$$

$$N_\zeta \equiv \zeta \bar{\zeta}, \quad (2.7)$$

$$N_{v^\oplus} \equiv v^\oplus \bar{v}^\oplus, \quad (2.8)$$

$$N_{v^\ominus} \equiv v^\ominus \bar{v}^\ominus, \quad (2.9)$$

$$N_v \equiv N_{v^\oplus} + N_{v^\ominus}, \quad (2.10)$$

and 2×2 matrices defined by

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \pi_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.11)$$

2.2 Global conformal symmetries

The conformal algebra $so(d, 2)$ of d dimensional space-time taken to be in basis of the Lorentz algebra $so(d-1, 1)$ consists of translation generators P^A , conformal boost generators K^A , and generators J^{AB} which span $so(d-1, 1)$ Lorentz algebra. We assume the following normalization for commutators of the conformal algebra:

$$[D, P^A] = -P^A, \quad [P^A, J^{BC}] = \eta^{AB} P^C - \eta^{AC} P^B, \quad (2.12)$$

$$[D, K^A] = K^A, \quad [K^A, J^{BC}] = \eta^{AB} K^C - \eta^{AC} K^B, \quad (2.13)$$

$$[P^A, K^B] = \eta^{AB} D - J^{AB}, \quad (2.14)$$

$$[J^{AB}, J^{CE}] = \eta^{BC} J^{AE} + 3 \text{ terms}. \quad (2.15)$$

⁷ We use oscillator formulation [22, 23, 24] to handle the many indices appearing for tensor fields. It can also be reformulated as an algebra acting on the symmetric-spinor bundle on the manifold M [25]. Note that the scalar oscillators ζ , $\bar{\zeta}$, which appeared in gauge invariant formulation of massive fields, arise naturally by a dimensional reduction [26, 25] from flat space. It is natural to expect that ‘conformal’ oscillators v^\oplus , v^\ominus , \bar{v}^\oplus , \bar{v}^\ominus also allow certain interpretation via dimensional reduction.

Let $|\phi\rangle$ denotes field propagating in flat space-time of dimension $d \geq 4$. Let Lagrangian for the free field $|\phi\rangle$ be conformal invariant. This implies, that Lagrangian is invariant with respect to transformation (invariance of the Lagrangian is assumed to be by module of total derivatives)

$$\delta_{\hat{G}}|\phi\rangle = \hat{G}|\phi\rangle, \quad (2.16)$$

where realization of the conformal algebra generators \hat{G} in terms of differential operators takes the form

$$P^A = \partial^A, \quad (2.17)$$

$$J^{AB} = x^A \partial^B - x^B \partial^A + M^{AB}, \quad (2.18)$$

$$D = x \partial + \Delta, \quad (2.19)$$

$$K^A = K_{\Delta, M}^A + R^A, \quad (2.20)$$

and we use the notation

$$K_{\Delta, M}^A \equiv -\frac{1}{2}x^2 \partial^A + x^A D + M^{AB} x^B, \quad (2.21)$$

$$x \partial \equiv x^A \partial^A, \quad x^2 = x^A x^A. \quad (2.22)$$

In (2.18)-(2.20), Δ is operator of conformal dimension, M^{AB} is spin operator of the Lorentz algebra,

$$[M^{AB}, M^{CE}] = \eta^{BC} M^{AE} + 3 \text{ terms}, \quad (2.23)$$

and R^A is operator depending on derivatives with respect to space-time coordinates and not depending on space-time coordinates x^A ,

$$[P^A, R^B] = 0. \quad (2.24)$$

The spin operator of the Lorentz algebra is well known for arbitrary spin conformal field. In standard, i.e. higher-derivative, formulations of conformal fields, the operator R^A is often equal to zero, while in ordinary-derivative approach, we develop in this paper, the operator R^A is non-trivial. This implies that complete description of the conformal fields in the ordinary-derivative approach requires finding not only gauge invariant Lagrangian but also the operator R^A as well. It turns out that requiring gauge invariance of Lagrangian and invariance with respect to global conformal symmetries allows us to fix both the Lagrangian and the operator R^A uniquely.

3 Conformal scalar field

As a warm up let us consider spin 0 field (scalar field). To make contact with studies in earlier literature we start with presentation of the standard, i.e. higher-derivative, formulation for the scalar field.

3.1 Higher-derivative formulation of conformal scalar field

In the framework of the standard approach conformal scalar field ϕ propagating in flat space of arbitrary dimension d is described by Lagrangian

$$\mathcal{L} = \frac{1}{2} \phi \square^{1+k} \phi, \quad \square \equiv \partial^A \partial^A, \quad (3.1)$$

where k is positive integer. For $k = 0$, the Lagrangian describes field associated with unitary representation of the conformal algebra $so(d, 2)$. For $k \geq 1$, the scalar field described by Lagrangian (3.1) turns out to be related to non-unitary representation of the conformal algebra⁸. The field ϕ has conformal dimension

$$\Delta_\phi = \frac{d-2}{2} - k. \quad (3.2)$$

3.2 Ordinary-derivative formulation of conformal scalar field

In the framework of ordinary-derivative approach, a dynamical system that on-shell is equivalent to the conformal scalar field ϕ with Lagrangian (3.1) and conformal dimension (3.2) involves $k + 1$ scalar fields

$$\phi_{0,k'}, \quad k' = -k, -k+2, \dots, k-2, k, \quad (3.3)$$

$$k - \text{arbitrary positive integer}. \quad (3.4)$$

Subscript 0 in $\phi_{0,k'}$ denotes Lorentz algebra spin, while the subscript k' determines conformal dimensions of the fields $\phi_{0,k'}$:

$$\Delta_{\phi_{0,k'}} = \frac{d-2}{2} + k'. \quad (3.5)$$

We note that, on-shell, the field $\phi_{0,-k}$ in (3.3) can be identified with the generic scalar field (3.1),

$$\phi = \phi_{0,-k}. \quad (3.6)$$

⁸ By now, representations of (super)conformal algebras that are relevant for elementary particles are well understood (for discussion of conformal algebras see e.g. [27]-[31] and superconformal algebras in [32, 33]). In contrast to this, non-unitary representation deserves to be understood better.

In order to obtain the Lagrangian description in an easy-to-use form, we introduce creation operators v^\oplus , v^\ominus and the respective annihilation operators \bar{v}^\ominus , \bar{v}^\oplus and use ket-vector defined by

$$|\phi\rangle \equiv \sum_{k'} (v^\oplus)^{\frac{k+k'}{2}} (v^\ominus)^{\frac{k-k'}{2}} \phi_{0,k'}(x) |0\rangle, \quad (3.7)$$

$$k' = -k, -k+2, \dots, k-2, k.$$

The ket-vector $|\phi\rangle$ (3.7) is immediately seen to be degree k homogeneous polynomial in the oscillators v^\oplus , v^\ominus ,

$$(N_v - k)|\phi\rangle = 0. \quad (3.8)$$

Ordinary-derivative Lagrangian can entirely be expressed in terms of the ket-vector $|\phi\rangle$. This is to say that Lagrangian we found takes the form⁹

$$\mathcal{L} = \frac{1}{2} \langle \phi | E | \phi \rangle, \quad (3.9)$$

where operator E is given by¹⁰

$$E = \square - m^2, \quad (3.10)$$

$$m^2 \equiv v^\ominus \bar{v}^\ominus. \quad (3.11)$$

Realization of conformal boost symmetries. To complete ordinary-derivative description of the scalar field we should provide realization of the conformal algebra symmetries on the space of the ket-vector $|\phi\rangle$. Largely, this realization for arbitrary spin field was already given in (2.17)-(2.20). All that is required is to fix operators M^{AB} , Δ and R^A for the particular case of the conformal scalar field. For the case of scalar field, the spin matrix of the Lorentz algebra is equal to zero, $M^{AB} = 0$. Realization of the operator of conformal dimension Δ on space $|\phi\rangle$ can be read from (3.5),

$$\Delta = \frac{d-2}{2} + \Delta', \quad (3.12)$$

$$\Delta' \equiv N_{v^\oplus} - N_{v^\ominus}. \quad (3.13)$$

Representation of the operator R^A on space of $|\phi\rangle$ is given by

$$R^A = -2v^\oplus \bar{v}^\oplus \partial^A. \quad (3.14)$$

⁹The bra-vector $\langle \phi |$ is defined according the rule $\langle \phi | = (|\phi\rangle)^\dagger$.

¹⁰ We introduce mass-like term m^2 to show formal similarity between our Lagrangian and the one for massive field.

Plugging this operator in (2.20) we make sure that conformal boost generator satisfies the expected commutator $[K^A, K^B] = 0$.

We note that, having been introduced field content, the Lagrangian and the operator R^A are fixed uniquely by requiring that¹¹

- i) Lagrangian should not involve higher than second order terms in derivatives;
- ii) the operator R^A should not involve higher than first order terms in derivatives;
- iii) Lagrangian should be invariant with respect to global conformal algebra symmetries.

4 Spin 1 conformal field

We proceed with discussion of conformal field theory for spin 1 field which is the simplest example allowing us to demonstrate how gauge symmetries are realized in the ordinary-derivative approach. As before, to make contact with studies in earlier literature we start with presentation of the standard, i.e. higher-derivative, formulation for the spin 1 field.

4.1 Higher-derivative formulation of spin 1 conformal field

In the framework of the standard approach, spin 1 conformal field ϕ^A propagating in flat space of arbitrary dimension $d \geq 4$ is described by Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{AB}\square^k F^{AB}, \quad F^{AB} \equiv \partial^A \phi^B - \partial^B \phi^A, \quad (4.1)$$

$$k \equiv \frac{d-4}{2}. \quad (4.2)$$

We see that for the case of spin 1 field the integer k turns out to be fixed by dimension of space-time. For $k = 0$ (i.e., $d = 4$), the Lagrangian (4.1) describes Maxwell vector field which is associated with unitary representation of the conformal algebra $so(4, 2)$. For $k \geq 1$ (i.e., $d \geq 6$), the spin 1 field, described by the Lagrangian (4.1), associates with non-unitary representation of the conformal algebra $so(d, 2)$. The field ϕ^A has conformal dimension independent of space-time dimension,

$$\Delta_{\phi^A} = 1. \quad (4.3)$$

Let us now discuss on-shell D.o.F of the conformal theory under consideration. For this purpose it is convenient to use fields transforming in irreps of the $so(d-2)$ algebra. Namely, we decompose on-shell D.o.F into irreps of the $so(d-2)$ algebra. One can prove (see Appendix A for details) that on-shell D.o.F are described by $k+1$ vector fields $\phi_{1,k'}^I$ and k scalar fields

¹¹ Various alternative discussions of higher-derivative theories may be found in [34, 35, 36].

$\phi_{0,k'}$:

$$\phi_{1,k'}^I \quad k' = -k, -k+2, \dots, k-2, k; \quad (4.4)$$

$$\phi_{0,k'}, \quad k' = -k+1, -k+3, \dots, k-3, k-1, \quad (4.5)$$

where vector indices of the $so(d-2)$ algebra take values $I = 1, 2, \dots, d-2$. The fields $\phi_{1,k'}^I$ and $\phi_{0,k'}$ transform in the respective vector and scalar representations of the $so(d-2)$ algebra. We note that the scalar on-shell D.o.F (4.5) appear only in $d \geq 6$ (i.e., $k \geq 1$). Total number of on-shell D.o.F given in (4.4),(4.5) is equal to

$$\nu = \frac{1}{2}d(d-3). \quad (4.6)$$

Namely, we note that ν is a sum of on-shell D.o.F for vector fields, $\nu(\phi_{1,k'})$, and on-shell D.o.F for scalar fields, $\nu(\phi_{0,k'})$, given in (4.4) and (4.5) respectively:

$$\nu = \sum_{k'} \nu(\phi_{1,k'}) + \sum_{k'} \nu(\phi_{0,k'}); \quad (4.7)$$

$$\nu(\phi_{1,k'}) = d-2, \quad k' = -k, -k+2, \dots, k-2, k, \quad (4.8)$$

$$\nu(\phi_{0,k'}) = 1, \quad k' = -k+1, -k+3, \dots, k-3, k-1. \quad (4.9)$$

4.2 Ordinary-derivative formulation of spin 1 conformal field

Field content. To discuss ordinary-derivative and gauge invariant formulation of spin 1 conformal field in flat space of dimension $d \geq 4$ we use $k+1$ vector fields $\phi_{1,k'}^A$ and k scalar fields $\phi_{0,k'}$:

$$\phi_{1,k'}^A \quad k' = -k, -k+2, \dots, k-2, k; \quad (4.10)$$

$$\phi_{0,k'}, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (4.11)$$

$$k \equiv \frac{d-4}{2}. \quad (4.12)$$

The fields $\phi_{1,k'}^A$ and $\phi_{0,k'}$ transform in the respective vector and scalar irreps of the Lorentz algebra $so(d-1, 1)$. Note that the scalar fields $\phi_{0,k'}$ (4.11) appear only in $d \geq 6$ (i.e. $k \geq 1$). Also, we note that the fields $\phi_{1,k'}^A$ and $\phi_{0,k'}$ (4.10),(4.11) have conformal dimensions

$$\Delta_{\phi_{1,k'}^A} = \frac{d-2}{2} + k', \quad \Delta_{\phi_{0,k'}} = \frac{d-2}{2} + k'. \quad (4.13)$$

Comparison of on-shell D.o.F given in (4.4),(4.5) and Lorentz fields (4.10),(4.11) demonstrates general rule to be used to obtain field content of gauge invariant ordinary-derivative

formulation. Namely, all that is required is to replace on-shell fields of the $so(d-2)$ algebra by the respective fields of the Lorentz algebra $so(d-1, 1)$ ¹².

In order to obtain the gauge invariant description in an easy-to-use form we use a set of the creation operators α^A , ζ , v^\oplus , v^\ominus , and the respective set of annihilation operators, $\bar{\alpha}^A$, $\bar{\zeta}$, \bar{v}^\ominus , \bar{v}^\oplus . The fields (4.10),(4.11) can then be collected into a ket-vector $|\phi\rangle$ defined by

$$|\phi\rangle = |\phi_1\rangle + \zeta|\phi_0\rangle, \quad (4.14)$$

$$|\phi_1\rangle \equiv \sum_{k'} \alpha^A (v^\oplus)^{\frac{k+k'}{2}} (v^\ominus)^{\frac{k-k'}{2}} \phi_{1,k'}^A(x) |0\rangle, \\ k' = -k, -k+2, \dots, k-2, k; \quad (4.15)$$

$$|\phi_0\rangle \equiv \sum_{k'} (v^\oplus)^{\frac{k-1+k'}{2}} (v^\ominus)^{\frac{k-1-k'}{2}} \phi_{0,k'}(x) |0\rangle, \\ k' = -k+1, -k+3, \dots, k-3, k-1. \quad (4.16)$$

From (4.14)-(4.16), we see that

a) the ket-vector $|\phi\rangle$ is degree 1 homogeneous polynomial in the oscillators α^A , ζ , i.e., $|\phi\rangle$ satisfies the relation

$$(N_\alpha + N_\zeta)|\phi\rangle = |\phi\rangle; \quad (4.17)$$

b) the ket-vector $|\phi\rangle$ is degree k homogeneous polynomial in the oscillators ζ , v^\oplus , v^\ominus ; i.e. $|\phi\rangle$ satisfies the relation

$$(N_\zeta + N_v)|\phi\rangle = k|\phi\rangle; \quad (4.18)$$

c) the ket-vectors $|\phi_1\rangle$ and $|\phi_0\rangle$ are the respective degree k and $k-1$ homogeneous polynomials in the oscillators v^\oplus , v^\ominus , i.e. $|\phi_1\rangle$ and $|\phi_0\rangle$ satisfy the relations

$$N_v|\phi_1\rangle = k|\phi_1\rangle, \quad N_v|\phi_0\rangle = (k-1)|\phi_0\rangle. \quad (4.19)$$

Below we shall see that it is the scalar fields (4.11) collected in $|\phi_0\rangle$ that are the Stueckelberg fields in the framework of the ordinary-derivative approach. Having described field content, we are ready to discuss Lagrangian in the framework of ordinary-derivative approach.

Lagrangian. Lagrangian we found takes the form

$$\mathcal{L} = \frac{1}{2} \langle \phi | E | \phi \rangle, \quad (4.20)$$

¹² Such a rule can be used when there is one-to-one mapping between spin labels of the $so(d-2)$ algebra and those of the Lorentz algebra $so(d-1, 1)$.

where operator E is defined by relations

$$E = E_{(2)} + E_{(1)} + E_{(0)}, \quad (4.21)$$

$$E_{(2)} = \square - \alpha \partial \bar{\alpha} \partial, \quad (4.22)$$

$$E_{(1)} = e_1 \alpha \partial - e_1^\dagger \bar{\alpha} \partial, \quad (4.23)$$

$$E_{(0)} = m_1, \quad (4.24)$$

$$e_1 = v^\ominus \bar{\zeta}, \quad (4.25)$$

$$e_1^\dagger = \zeta \bar{v}^\ominus, \quad (4.26)$$

$$m_1 = v^\ominus \bar{v}^\ominus (N_\zeta - 1). \quad (4.27)$$

We note that $E_{(2)}$ (4.22) is standard second-order Maxwell operator rewritten in terms of the oscillators.

Gauge transformations. We now discuss gauge symmetries of the conformal theory under consideration. To this end we introduce gauge transformation parameters

$$\epsilon_{0,k'-1} \quad k' = -k, -k+2, \dots, k-2, k. \quad (4.28)$$

The gauge transformation parameters $\epsilon_{0,k'}$ are scalar fields of the Lorentz algebra $so(d-1, 1)$. As usually, we collect $\epsilon_{0,k'}$ in ket-vector $|\epsilon\rangle$ defined by

$$|\epsilon\rangle \equiv \sum_{k'} (v^\oplus)^{\frac{k+k'}{2}} (v^\ominus)^{\frac{k-k'}{2}} \epsilon_{0,k'-1}(x) |0\rangle, \quad (4.29)$$

$$k' = -k, -k+2, \dots, k-2, k.$$

The ket-vector of gauge transformation parameters $|\epsilon\rangle$ satisfies the algebraic constraint

$$N_v |\epsilon\rangle = k |\epsilon\rangle, \quad (4.30)$$

which tells us that $|\epsilon\rangle$ is a degree k homogeneous polynomial in the oscillators v^\oplus, v^\ominus .

In terms of $|\phi\rangle$ and $|\epsilon\rangle$, gauge transformations take the form

$$\delta |\phi\rangle = (\alpha \partial + b_1) |\epsilon\rangle, \quad (4.31)$$

$$b_1 = \zeta \bar{v}^\ominus. \quad (4.32)$$

Two remarks are in order.

i) Making use of (4.14),(4.29) it is easy to see that gauge transformations for the ket-vector (4.31) lead to the following gauge transformation for the component fields:

$$\delta\phi_{1,k'}^A \sim \partial^A \epsilon_{0,k'-1} , \quad (4.33)$$

$$\delta\phi_{0,k'} \sim \epsilon_{0,k'} . \quad (4.34)$$

From (4.34), we see that all the scalar fields $\phi_{0,k'}$ (4.11) can be gauged away, i.e. these scalar fields are nothing but the Stueckelberg fields in the framework of the ordinary-derivative approach. Thus, we see that our Stueckelberg fields are similar to those used for gauge invariant formulation of massive spin 1 field, i.e. all the Stueckelberg fields can be gauged away.

ii) There is some difference as compared to gauge formulation for massive field. For the case of massive spin 1 field, number of gauge transformation parameters is equal to number of Stueckelberg fields. In our case, number of gauge transformation parameters is greater than number of the Stueckelberg fields. This is to say that we have $k + 1$ gauge transformation parameters (see (4.28)) and k Stueckelberg fields (see (4.11)). This implies that having gauged away the Stueckelberg fields we still have one surviving gauge symmetry that is generated by gauge parameter $\epsilon_{0,-k-1}$. This surviving gauge symmetry is nothing but the gauge symmetry of the generic higher-derivative formulation (4.1).

Realization of conformal boost symmetries. To complete ordinary-derivative formulation we provide realization of the conformal algebra symmetries on space of the ket-vector $|\phi\rangle$. All that is required is to fix operators M^{AB} , Δ and R^A for the case of spin 1 conformal field and then use relations given in (2.17)-(2.20). For the case of spin 1 field, the spin matrix of the Lorentz algebra takes the form

$$M^{AB} = \alpha^A \bar{\alpha}^B - \alpha^B \bar{\alpha}^A . \quad (4.35)$$

Realization of the operator of conformal dimension Δ on space of $|\phi\rangle$ can be read from (4.13),

$$\Delta = \frac{d-2}{2} + \Delta' , \quad (4.36)$$

$$\Delta' \equiv N_{v^\oplus} - N_{v^\ominus} . \quad (4.37)$$

Representation of the operator R^A on space of $|\phi\rangle$ is given by

$$R^A = R_0^A + R_1^A + R_G^A, \quad (4.38)$$

$$R_0^A = r_{0,1}\alpha^A - r_{0,1}^\dagger\bar{\alpha}^A, \quad (4.39)$$

$$R_1^A = r_{1,1}\partial^A, \quad (4.40)$$

$$R_G^A = Gr_G^A, \quad r_G^A = r_{G,1}\bar{\alpha}^A, \quad (4.41)$$

$$r_{0,1} = 2v^\oplus\bar{\zeta}, \quad (4.42)$$

$$r_{0,1}^\dagger = 2\zeta\bar{v}^\oplus, \quad (4.43)$$

$$r_{1,1} = -2v^\oplus\bar{v}^\oplus, \quad (4.44)$$

$$r_{G,1} = v^\oplus\tilde{r}_{G,1}\bar{v}^\oplus, \quad \tilde{r}_{G,1} = \tilde{r}_{G,1}(\Delta'), \quad (4.45)$$

where G (4.41) stands for operator of gauge transformation (4.31), $G = \alpha\partial + b_1$, and $\tilde{r}_{G,1}$ (4.45) is arbitrary function of the operator Δ' .

Two remarks are in order.

i) From (4.38)-(4.45), we see that R_0^A and R_1^A parts of the operator R^A are fixed uniquely, while R_G^A part, in view of arbitrary $\tilde{r}_{G,1}$, is still to be arbitrary. Reason for arbitrariness in R_G^A is obvious. Global transformations of gauge fields are defined by module of gauge transformations. Because of R_G^A is proportional to gauge transformation operator G , action of R_G^A on gauge field takes the form of some special gauge transformation.

ii) Evaluating commutator $[K^A, K^B]$, we obtain $[K^A, K^B] \sim Gr_G^{AB}$, where r_G^{AB} is some differential operator, i.e. commutator of conformal boost generator K^A is proportional to the operator of gauge transformation, as it should in gauge theory. If we impose requirement $[K^A, K^B] = 0$, which amounts to $r_G^{AB} = 0$, then we obtain equations for $\tilde{r}_{G,1}$. Solution to these equations takes the form

$$\tilde{r}_{G,1} = \frac{4}{\Delta' + c_0}, \quad c_0 \neq -k+1, -k+3, \dots, k-3, k-1, \quad (4.46)$$

where c_0 is constant. Condition on c_0 in (4.46) is obtained by requiring that the operator R_G^A be well defined when acting on the ket-vector $|\phi\rangle$ (4.14). Note that the simplest representative $R_G^A = 0$ is achieved by taking $c_0 = \infty$.

We summarize our result in this section. We started with field content implemented by analysis of on-shell D.o.F of higher-derivative spin 1 conformal theory and look for ordinary-derivative formulation by requiring that:

i) Lagrangian should not involve higher than second order terms in derivatives.

- ii) gauge transformations and operator R^A should not involve higher than first order terms in derivatives.
- iii) Lagrangian should be invariant with respect to gauge transformations and global conformal algebra transformations.

These requirements fix Lagrangian and gauge transformations uniquely. The operator R^A is fixed uniquely by module of gauge transformation operator, as it is expected in any theory of gauge fields.

We finish discussion of ordinary-derivative formulation of spin 1 conformal theory by presenting component form of the Lagrangian and the gauge transformations for the case of spin 1 conformal field in $d = 6$ (i.e. $k = 1$). This is the simplest example involving the Stueckelberg fields.

Spin 1 conformal field in $d = 6$. For this case, our approach involves two vector fields $\phi_{1,1}^A$, $\phi_{1,-1}^A$ and one scalar Stueckelberg field $\phi_{0,0}$ (see (4.10),(4.11)). In terms of these fields, the Lagrangian (4.20) takes the form

$$\begin{aligned} \mathcal{L} = & \phi_{1,1}^A (\eta^{AB} \square - \partial^A \partial^B) \phi_{1,-1}^B + \frac{1}{2} \phi_{0,0} \square \phi_{0,0} \\ & - \phi_{0,0} \partial^A \phi_{1,1}^A - \frac{1}{2} \phi_{1,1}^A \phi_{1,1}^A . \end{aligned} \quad (4.47)$$

This Lagrangian is invariant with respect to gauge transformations (4.31),

$$\delta \phi_{1,-1}^A = \partial^A \epsilon_{0,-2} , \quad (4.48)$$

$$\delta \phi_{1,1}^A = \partial^A \epsilon_{0,0} , \quad (4.49)$$

$$\delta \phi_{0,0} = \epsilon_{0,0} . \quad (4.50)$$

From (4.50), we see that the scalar field $\phi_{0,0}$ is indeed the Stueckelberg field. Using E.o.M for the field $\phi_{1,1}^A$ allows us to solve the field $\phi_{1,-1}^A$ in terms of the remaining fields,

$$\phi_{1,1}^A = (\eta^{AB} \square - \partial^A \partial^B) \phi_{1,-1}^B - \partial^A \phi_{0,0} . \quad (4.51)$$

Plugging this solution in Lagrangian (4.47) gives the standard higher-derivative Lagrangian (4.1) for $d = 6$,

$$\mathcal{L} = \frac{1}{2} \phi_{1,-1}^A \square (\eta^{AB} \square - \partial^A \partial^B) \phi_{1,-1}^B . \quad (4.52)$$

5 Spin 2 conformal field

We proceed our study of ordinary-derivative formulation of conformal field theory with discussion of spin 2 conformal field. In the literature, such field is often referred to as conformal

Weyl graviton. As before, to make contact with studies in earlier literature we start with presentation of the standard, i.e. higher-derivative, formulation for the spin 2 conformal field. In due course, we review our result concerning the counting of on-shell D.o.F for spin 2 conformal field.

5.1 Higher-derivative formulation of spin 2 conformal field

In the framework of the standard approach, spin 2 conformal field ϕ^{AB} propagating in flat space of arbitrary dimension $d \geq 4$ is described by Lagrangian

$$\mathcal{L} = \frac{1}{a^2} C^{ABCE} \square^{k-1} C^{ABCE}, \quad a^2 \equiv 4 \frac{d-3}{d-2}, \quad (5.1)$$

$$k \equiv \frac{d-2}{2}, \quad (5.2)$$

where C^{ABCE} is the Weyl tensor. The field ϕ^{AB} has conformal dimension $\Delta_{\phi^{AB}} = 0$. Lagrangian (5.1) can be rewritten as

$$\mathcal{L} = \frac{1}{4} \phi_{AB} \square^{k+1} P_{A'B'}^{AB} \phi^{A'B'}, \quad (5.3)$$

and we use notation as in [1]:

$$P_{A'B'}^{AB} \equiv \pi_{(A'}^A \pi_{B')}^B - \frac{1}{d-1} \pi^{AB} \pi_{A'B'}, \quad (5.4)$$

$$\pi^{AB} \equiv \eta^{AB} - \frac{\partial^A \partial^B}{\square}. \quad (5.5)$$

Lagrangian (5.3) is invariant with respect to linearized diffeomorphism and local Weyl gauge transformations

$$\delta \phi^{AB} = \partial^A \xi^B + \partial^B \xi^A, \quad (5.6)$$

$$\delta \phi^{AB} = \eta^{AB} \xi, \quad (5.7)$$

where ξ^A and ξ are the respective diffeomorphisms and Weyl gauge transformation parameters.

We now discuss on-shell D.o.F of the conformal theory under consideration. As before, to discuss on-shell D.o.F we use fields transforming in irreps of $so(d-2)$ algebra. One can prove (see Appendix B for details) that on-shell D.o.F are described by $k+1$ rank-2 traceless symmetric tensor fields $\phi_{1,k'}^{IJ}$, k vector fields $\phi_{1,k'}^I$, and $k-1$ scalar fields $\phi_{0,k'}$:

$$\phi_{2,k'}^{IJ}, \quad k' = -k, -k+2, \dots, k-2, k; \quad (5.8)$$

$$\phi_{1,k'}^I, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (5.9)$$

$$\phi_{0,k'}, \quad k' = -k+2, -k+4, \dots, k-4, k-2, \quad (5.10)$$

$I, J = 1, \dots, d-2$. Note that scalar on-shell D.o.F (5.10) appear in spin 2 conformal theories only in $d \geq 6$ (i.e. $k \geq 2$). Total number of on-shell D.o.F shown in (5.8)-(5.10) is given by

$$\nu = \frac{1}{4}d(d-3)(d+2). \quad (5.11)$$

We note that this ν is a sum of D.o.F for fields given in (5.8)-(5.10)¹³:

$$\nu = \sum_{k'} \nu(\phi_{2,k'}) + \sum_{k'} \nu(\phi_{1,k'}) + \sum_{k'} \nu(\phi_{0,k'}), \quad (5.12)$$

$$\nu(\phi_{2,k'}) = \frac{d(d-3)}{2}, \quad k' = -k, -k+2, \dots, k-2, k; \quad (5.13)$$

$$\nu(\phi_{1,k'}) = d-2, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (5.14)$$

$$\nu(\phi_{0,k'}) = 1, \quad k' = -k+2, -k+4, \dots, k-4, k-2. \quad (5.15)$$

For various dimensions, the ν and the decomposition (5.12) are as follows:

| | | | |
|----------|-------------|--|--------|
| $d = 4$ | $\nu = 6$ | $2 \times \mathbf{2}_2 + 1 \times \mathbf{2}_1 + 0 \times \mathbf{1}_0$ | |
| $d = 6$ | $\nu = 36$ | $3 \times \mathbf{9}_2 + 2 \times \mathbf{4}_1 + 1 \times \mathbf{1}_0$ | |
| $d = 8$ | $\nu = 100$ | $4 \times \mathbf{20}_2 + 3 \times \mathbf{6}_1 + 2 \times \mathbf{1}_0$ | |
| $d = 10$ | $\nu = 210$ | $5 \times \mathbf{35}_2 + 4 \times \mathbf{8}_1 + 3 \times \mathbf{1}_0$ | (5.16) |

In last column in (5.16) in expressions like $X \times \mathbf{Y}_Z$, the \mathbf{Y} stands for dimension of spin Z totally symmetric irreps of the $so(d-2)$ algebra, while X stands for multiplicity of the spin Z irreps of the $so(d-2)$ algebra.

5.2 Ordinary-derivative formulation of spin 2 conformal field

Field content. To discuss ordinary-derivative and gauge invariant formulation of spin 2 conformal field in flat space of dimension $d \geq 4$ we use $k+1$ rank-2 symmetric tensor fields $\phi_{2,k'}^{AB}$, k vector fields $\phi_{1,k'}^A$, and $k-1$ scalar fields $\phi_{0,k'}$:

$$\phi_{2,k'}^{AB}, \quad k' = -k, -k+2, \dots, k-2, k; \quad (5.17)$$

$$\phi_{1,k'}^A, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (5.18)$$

$$\phi_{0,k'}, \quad k' = -k+2, -k+4, \dots, k-4, k-2; \quad (5.19)$$

¹³ Total D.o.F given in (5.11) was found in [1]. Decomposition of ν (5.12) into irreps of the $so(d-2)$ algebra for the case of $d = 4$ spin 2 conformal field theory was carried out in [37]. In Appendix B, we use light-cone approach to generalize result of the latter reference to case of arbitrary $d \geq 4$.

$$k \equiv \frac{d-2}{2}. \quad (5.20)$$

The fields $\phi_{2,k'}^{AB}$, $\phi_{1,k'}^A$ and $\phi_{0,k'}$ are the respective rank-2 tensor, vector, and scalar fields of the Lorentz algebra $so(d-1, 1)$. Note that the scalar fields (5.19) appear only in $d \geq 6$ (i.e. $k \geq 2$). Also, we note that the fields in (5.17)-(5.19) have conformal dimensions

$$\Delta_{\phi_{2,k'}^{AB}} = \frac{d-2}{2} + k', \quad \Delta_{\phi_{1,k'}^A} = \frac{d-2}{2} + k', \quad \Delta_{\phi_{0,k'}} = \frac{d-2}{2} + k'. \quad (5.21)$$

As before, we use the oscillators α^A , ζ , v^\oplus , v^\ominus to collect the fields (5.17)-(5.19) into a ket-vector $|\phi\rangle$ defined by

$$|\phi\rangle = |\phi_2\rangle + \zeta|\phi_1\rangle + \zeta^2|\phi_0\rangle, \quad (5.22)$$

$$|\phi_2\rangle \equiv \sum_{k'} \alpha^A \alpha^B (v^\oplus)^{\frac{k+k'}{2}} (v^\ominus)^{\frac{k-k'}{2}} \phi_{2,k'}^{AB}(x) |0\rangle, \quad (5.23)$$

$$k' = -k, -k+2, \dots, k-2, k;$$

$$|\phi_1\rangle \equiv \sum_{k'} \alpha^A (v^\oplus)^{\frac{k-1+k'}{2}} (v^\ominus)^{\frac{k-1-k'}{2}} \phi_{1,k'}^A(x) |0\rangle, \quad (5.24)$$

$$k' = -k+1, -k+3, \dots, k-3, k-1;$$

$$|\phi_0\rangle \equiv \sum_{k'} (v^\oplus)^{\frac{k-2+k'}{2}} (v^\ominus)^{\frac{k-2-k'}{2}} \phi_{0,k'}(x) |0\rangle, \quad (5.25)$$

$$k' = -k+2, -k+4, \dots, k-4, k-2.$$

From (5.22)-(5.25), we see that the ket-vector $|\phi\rangle$ is degree 2 homogeneous polynomial in the oscillators α^A , ζ and degree k homogeneous polynomial in the oscillators ζ , v^\oplus , v^\ominus . In other words, the ket-vector $|\phi\rangle$ satisfies the relations

$$(N_\alpha + N_\zeta)|\phi\rangle = 2|\phi\rangle, \quad (5.26)$$

$$(N_\zeta + N_v)|\phi\rangle = k|\phi\rangle. \quad (5.27)$$

Also, note that the ket-vectors $|\phi_2\rangle$, $|\phi_1\rangle$, and $|\phi_0\rangle$ are the respective degree k , $k-1$, and $k-2$ homogeneous polynomials in the oscillators v^\oplus , v^\ominus , i.e., these ket-vectors satisfy the relations

$$N_v|\phi_2\rangle = k|\phi_2\rangle, \quad N_v|\phi_1\rangle = (k-1)|\phi_1\rangle, \quad N_v|\phi_0\rangle = (k-2)|\phi_0\rangle. \quad (5.28)$$

We note that ket-vector $|\phi_0\rangle$, which collects the scalar fields (5.19), appear only in $d \geq 6$ (i.e. $k \geq 2$).

Lagrangian. Lagrangian we found takes the form

$$\mathcal{L} = \frac{1}{2} \langle \phi | E | \phi \rangle, \quad (5.29)$$

where operator E is given by

$$E = E_{(2)} + E_{(1)} + E_{(0)}, \quad (5.30)$$

$$E_{(2)} \equiv \square - \alpha \partial \bar{\alpha} \partial + \frac{1}{2} (\alpha \partial)^2 \bar{\alpha}^2 + \frac{1}{2} \alpha^2 (\bar{\alpha} \partial)^2 - \frac{1}{2} \alpha^2 \square \bar{\alpha}^2, \quad (5.31)$$

$$E_{(1)} \equiv e_1 (\alpha \partial - \alpha^2 \bar{\alpha} \partial) - e_1^\dagger (\bar{\alpha} \partial - \alpha \partial \bar{\alpha}^2), \quad (5.32)$$

$$E_{(0)} \equiv m_1 + \alpha^2 \bar{\alpha}^2 m_2 + m_3 \alpha^2 + m_3^\dagger \bar{\alpha}^2, \quad (5.33)$$

$$e_1 = v^\ominus \left(\frac{d - N_\zeta}{d - 2N_\zeta} \right)^{1/2} \bar{\zeta}, \quad (5.34)$$

$$e_1^\dagger = \zeta \left(\frac{d - N_\zeta}{d - 2N_\zeta} \right)^{1/2} \bar{v}^\ominus, \quad (5.35)$$

$$m_1 = v^\ominus \bar{v}^\ominus \frac{d + 2 - N_\zeta}{d + 2 - 2N_\zeta} (N_\zeta - 1), \quad (5.36)$$

$$m_2 = \frac{1}{2} v^\ominus \bar{v}^\ominus, \quad (5.37)$$

$$m_3 = \frac{1}{2} \left(\frac{d - 1}{d - 2} \right)^{1/2} v^\ominus v^\ominus \bar{\zeta}^2, \quad (5.38)$$

$$m_3^\dagger = \frac{1}{2} \left(\frac{d - 1}{d - 2} \right)^{1/2} \zeta^2 \bar{v}^\ominus \bar{v}^\ominus. \quad (5.39)$$

We note that $E_{(2)}$ (5.31) is standard second-order Einstein-Hilbert operator rewritten in terms of the oscillators.

Gauge transformations. We now discuss gauge symmetries of the Lagrangian. To this end we introduce gauge transformation parameters,

$$\epsilon_{1,k'-1}^A, \quad k' = -k, -k + 2, \dots, k - 2, k; \quad (5.40)$$

$$\epsilon_{0,k'-1}, \quad k' = -k + 1, -k + 3, \dots, k - 3, k - 1. \quad (5.41)$$

The gauge transformation parameters $\epsilon_{1,k'}^A$ and $\epsilon_{0,k'}$ are the respective vector and scalar fields of the Lorentz algebra $so(d-1, 1)$. Then, as usually, we collect gauge transformation parameters in ket-vector $|\epsilon\rangle$ defined by

$$|\epsilon\rangle = |\epsilon_1\rangle + \zeta|\epsilon_0\rangle, \quad (5.42)$$

$$|\epsilon_1\rangle \equiv \sum_{k'} \alpha^A (v^\oplus)^{\frac{k+k'}{2}} (v^\ominus)^{\frac{k-k'}{2}} \epsilon_{1,k'-1}^A(x) |0\rangle, \\ k' = -k, -k+2, \dots, k-2, k; \quad (5.43)$$

$$|\epsilon_0\rangle \equiv \sum_{k'} (v^\oplus)^{\frac{k-1+k'}{2}} (v^\ominus)^{\frac{k-1-k'}{2}} \epsilon_{0,k'-1}(x) |0\rangle, \\ k' = -k+1, -k+3, \dots, k-3, k-1. \quad (5.44)$$

The ket-vector $|\epsilon\rangle$ satisfies the algebraic constraints

$$(N_\alpha + N_\zeta)|\epsilon\rangle = |\epsilon\rangle, \quad (5.45)$$

$$(N_\zeta + N_v)|\epsilon\rangle = k|\epsilon\rangle, \quad (5.46)$$

which tell us that $|\epsilon\rangle$ is a degree 1 homogeneous polynomial in the oscillators α^A, ζ and degree k homogeneous polynomial in the oscillators $\zeta, v^\oplus, v^\ominus$. The ket-vectors $|\epsilon_1\rangle$ and $|\epsilon_0\rangle$ satisfy the algebraic constraints

$$N_v|\phi_1\rangle = k|\phi_1\rangle, \quad N_v|\phi_0\rangle = (k-1)|\phi_0\rangle, \quad (5.47)$$

which imply that $|\epsilon_1\rangle$ and $|\epsilon_0\rangle$ are the respective degree k and $k-1$ homogeneous polynomials in the oscillators v^\oplus, v^\ominus .

Gauge transformations can entirely be written in terms of $|\phi\rangle$ and $|\epsilon\rangle$. This is to say that gauge transformations take the form

$$\delta|\phi\rangle = (\alpha\partial + b_1 + b_2\alpha^2)|\epsilon\rangle, \quad (5.48)$$

$$b_1 \equiv \zeta \left(\frac{d - N_\zeta}{d - 2N_\zeta} \right)^{1/2} \bar{v}^\ominus, \quad (5.49)$$

$$b_2 \equiv -\frac{1}{d-2} v^\ominus \bar{\zeta}. \quad (5.50)$$

Two remarks are in order.

Using (5.22),(5.42) one can make sure that the gauge transformations (5.48) lead to the following gauge transformations for the component fields:

$$\delta\phi_{2,k'}^{AB} \sim \partial^A \epsilon_{1,k'-1}^B + \partial^B \epsilon_{1,k'-1}^A + \eta^{AB} \epsilon_{0,k'} , \quad (5.51)$$

$$\delta\phi_{1,k'}^A \sim \partial^A \epsilon_{0,k'-1} + \epsilon_{1,k'}^A , \quad (5.52)$$

$$\delta\phi_{0,k'} \sim \epsilon_{0,k'} . \quad (5.53)$$

From (5.52),(5.53), we see that all the vector fields $\phi_{1,k'}^A$ and the scalar fields $\phi_{0,k'}$ can be gauged away, i.e. these fields are nothing but the Stueckelberg fields in the framework of the ordinary-derivative approach. Thus, we see that all Stueckelberg fields of ordinary-derivative-approach can be gauged away, as in the case of gauge invariant formulation of massive spin 2 field.

ii) However there is some difference as compared to gauge formulation for massive spin 2 field. For the case of massive spin 2 field, number of gauge transformation parameters is equal to number of Stueckelberg fields. In our case, the number of gauge transformation parameters is greater than the number of Stueckelberg fields. This is to say that, firstly, we have $k+1$ vector gauge transformation parameters (see (5.40)) and k Stueckelberg vector fields (see (5.18)), secondly, we have k scalar gauge transformation parameters (see (5.41)) and $k-1$ Stueckelberg scalar fields (see (5.19)). This implies that having gauged away the Stueckelberg vector fields and Stueckelberg scalar fields we still have one surviving gauge symmetry that is generated by vector gauge transformation parameter $\epsilon_{1,-k-1}^A$ and one surviving gauge symmetry that is generated by scalar gauge transformation parameter $\epsilon_{0,-k}$. These two surviving gauge symmetries are nothing but the gauge symmetries of the generic higher-derivative formulation (5.6),(5.7).

Realization of conformal boost symmetries. To complete ordinary-derivative formulation of spin 2 field we provide realization of the conformal algebra symmetries on the space of the ket-vector $|\phi\rangle$. All that is required is to fix operators M^{AB} , Δ and R^A for the case of spin 2 conformal field and then use these operators in (2.17)-(2.20). For the case of spin 2 field the spin matrix of the Lorentz algebra takes the form

$$M^{AB} = \alpha^A \bar{\alpha}^B - \alpha^B \bar{\alpha}^A . \quad (5.54)$$

Realization of the operator of conformal dimension Δ on space of $|\phi\rangle$ can be read from (5.21),

$$\Delta = \frac{d-2}{2} + \Delta' , \quad (5.55)$$

$$\Delta' \equiv N_{v^\oplus} - N_{v^\ominus} . \quad (5.56)$$

Representation of the operator R^A on space of $|\phi\rangle$ is given by

$$R^A = R_0^A + R_1^A + R_G^A, \quad (5.57)$$

$$R_0^A = r_{0,1}(\alpha^A - \frac{1}{d-2}\alpha^2\bar{\alpha}^A) - r_{0,1}^\dagger\bar{\alpha}^A, \quad (5.58)$$

$$R_1^A = r_{1,1}\partial^A, \quad (5.59)$$

$$R_G^A = Gr_G^A, \quad (5.60)$$

$$r_G^A = r_{G,1}\bar{\alpha}^A + r_{G,2}\alpha^A + r_{G,3}\alpha^A\bar{\alpha}^2, \quad (5.61)$$

$$r_{0,1} = 2v^\oplus \left(\frac{d - N_\zeta}{d - 2N_\zeta} \right)^{1/2} \bar{\zeta}, \quad (5.62)$$

$$r_{0,1}^\dagger = 2\zeta \left(\frac{d - N_\zeta}{d - 2N_\zeta} \right)^{1/2} \bar{v}^\oplus, \quad (5.63)$$

$$r_{1,1} = -2v^\oplus \bar{v}^\oplus, \quad (5.64)$$

$$r_{G,a} = v^\oplus \tilde{r}_{G,a} \bar{v}^\oplus, \quad a = 1, 3; \quad (5.65)$$

$$r_{G,a} = v^\oplus v^\oplus \tilde{r}_{G,a} \bar{\zeta}^2, \quad a = 2 \quad (5.66)$$

$$\begin{aligned} \tilde{r}_{G,a} &= \tilde{r}_{G,a}(0, \Delta')(1 - N_\zeta) \\ &\quad + \tilde{r}_{G,a}(1, \Delta')N_\zeta, \quad a = 1 \end{aligned} \quad (5.67)$$

$$\tilde{r}_{G,a} = \tilde{r}_{G,a}(\Delta'), \quad a = 2, 3, \quad (5.68)$$

where G (5.60) stands for operator of gauge transformation (5.48), $G = \alpha\partial + b_1 + b_2\alpha^2$, and $\tilde{r}_{G,1}, \tilde{r}_{G,2}, \tilde{r}_{G,3}$ (4.45) are arbitrary functions of the operator Δ' .

Two remarks are in order.

i) R_0^A and R_1^A parts of the operator R^A are fixed uniquely, while R_G^A part, in view of arbitrary $\tilde{r}_{G,1}, \tilde{r}_{G,2}, \tilde{r}_{G,3}$, is still to be arbitrary. Reason for arbitrariness in R_G^A is obvious. Global transformations of gauge fields are defined by module of gauge transformations. Since R_G^A is proportional to G , action of R_G^A on gauge field takes the form of some special gauge transformation.

ii) Evaluating commutator $[K^A, K^B]$, we obtain $[K^A, K^B] \sim Gr_G^{AB}$, where r_G^{AB} is some differential operator. In other words, $[K^A, K^B]$ is proportional to the operator of gauge transformation, as it should in gauge theory. If we impose requirement $[K^A, K^B] = 0$, which amounts to $r_G^{AB} = 0$, then we obtain equations for $\tilde{r}_{G,a}$. There are two solutions to these equations:

1st solution:

$$\tilde{r}_{G,1}(0, \Delta') = \frac{4}{\Delta' + x_{1,0}}, \quad x_{1,0} \neq -k+1, -k+3, \dots, k-3, k-1, \quad (5.69)$$

$$\tilde{r}_{G,1}(1, \Delta') = \frac{4}{\Delta' + x_{1,1}}, \quad x_{1,1} \neq -k, -k+2, \dots, k-2, k, \quad (5.70)$$

$$\tilde{r}_{G,2} = \frac{x_2}{\Delta' + 1 + x_{1,0}}, \quad \tilde{r}_{G,3} = 0, \quad (5.71)$$

$$x_2 = \frac{2}{\tilde{b}_1(1)k(k - x_{1,1})}((k+1)x_{1,1} - kx_{1,0}). \quad (5.72)$$

2nd solution:

$$\tilde{r}_{G,1}(0, \Delta') = 0, \quad (5.73)$$

$$\tilde{r}_{G,1}(1, \Delta') = \frac{4}{\Delta' + y_{1,1}}, \quad y_{1,1} \neq -k, -k+2, \dots, k-2, k, \quad (5.74)$$

$$\tilde{r}_{G,2} = \frac{y_2}{\Delta' + 1 + y_3}, \quad y_3 \neq -k+1, -k+3, \dots, k-3, k-1, \quad (5.75)$$

$$\tilde{r}_{G,3} = -\frac{2}{\Delta' + y_3}, \quad (5.76)$$

$$y_2 = \frac{2}{\tilde{b}_1(1)k(k - y_{1,1})}(y_{1,1} + ky_3 + k(k+2)). \quad (5.77)$$

In (5.69)-(5.75), $x_{1,0}$, $x_{1,1}$, $y_{1,1}$, y_3 are constants and $\tilde{b}_1(1) = (\frac{d-1}{d-2})^{1/2}$. Conditions on these constants (see (5.69), (5.70), (5.74), (5.75)) are obtained by requiring that the operator R_G^A be well defined when acting on the ket-vector $|\phi\rangle$ (5.22). Note that the simplest representative $R_G^A = 0$ is achieved by taking $x_{0,1} = x_{1,1} = \infty$ and $y_{1,1} = y_3 = \infty$.

We finish discussion of ordinary-derivative formulation of spin 2 conformal theory by presenting component form of the Lagrangian and the gauge transformations for the case of spin 2 conformal field in $d = 4$ (i.e. $k = 1$). This is the simplest conformal theory of spin 2 field involving the Stueckelberg field.

Spin 2 conformal field in $d = 4$. For this case, our approach involves two rank-2 tensor fields $\phi_{2,1}^{AB}$, $\phi_{2,-1}^{AB}$ and one vector Stueckelberg field $\phi_{1,0}^A$ (see (5.17),(5.18)). In terms of these

fields, the Lagrangian (5.29) takes the form¹⁴

$$\begin{aligned}
\mathcal{L} = & \frac{1}{2}\phi_{2,1}^{AB}(E_{EH}\phi_{2,-1})^{AB} + \frac{1}{2}\phi_{1,0}^A(E_{Max}\phi_{1,0})^A \\
& + \frac{1}{2}\phi_{2,1}^{AB}(\partial^A\phi_{1,0}^B + \partial^B\phi_{1,0}^A - 2\eta^{AB}\partial^C\phi_{1,0}^C) \\
& - \frac{1}{4}\phi_{2,1}^{AB}\phi_{2,1}^{AB} + \frac{1}{4}\phi_{2,1}^{AA}\phi_{2,1}^{BB}, \tag{5.78}
\end{aligned}$$

where the second-derivative Einstein-Hilbert and Maxwell operators E_{EH} , E_{Max} are given by

$$\begin{aligned}
(E_{EH}\phi)^{AB} &= \square\phi^{AB} - \partial^A(\partial\phi)^B - \partial^B(\partial\phi)^A + \partial^A\partial^B\phi + \eta^{AB}(\partial\partial\phi - \square\phi), \\
(\partial\phi)^A &\equiv \partial^B\phi^{AB}, \quad \partial\partial\phi \equiv \partial^A\partial^B\phi^{AB}, \quad \phi \equiv \phi^{AA}, \tag{5.79}
\end{aligned}$$

$$(E_{Max}\phi)^A = \square\phi^A - \partial^A\partial^B\phi^B. \tag{5.80}$$

In terms of component fields, the gauge transformations (5.48) take the form

$$\delta\phi_{2,-1}^{AB} = \partial^A\epsilon_{1,-2}^B + \partial^B\epsilon_{1,-2}^A - \eta^{AB}\epsilon_{0,-1} \tag{5.81}$$

$$\delta\phi_{2,1}^{AB} = \partial^A\epsilon_{1,0}^B + \partial^B\epsilon_{1,0}^A, \tag{5.82}$$

$$\delta\phi_{1,0}^A = \partial^A\epsilon_{0,-1} + \epsilon_{1,0}^A. \tag{5.83}$$

From (5.83), we see that vector field $\phi_{1,0}^A$ is the Stueckelberg field. Variation of action with respect to the field $\phi_{2,1}^{AB}$ gives E.o.M which allow us to solve $\phi_{2,1}^{AB}$ in terms of the remaining fields $\phi_{2,-1}^{AB}$ and $\phi_{1,0}^A$. Plugging solution for $\phi_{2,1}^{AB}$ in Lagrangian gives the standard higher-derivative Lagrangian (5.3) in terms of the conformal graviton field $\phi^{AB} \equiv \phi_{2,-1}^{AB}$.

We note that, as in case of spin 1 conformal theory, we derived the Lagrangian, gauge transformations, and operator R^A by imposing the following requirements:

- i) Lagrangian should not involve higher than *second order* terms in derivatives.
- ii) gauge transformations and operator R^A should not involve higher than *first order* terms in derivatives.
- iii) Lagrangian should be invariant with respect to gauge transformations and global conformal algebra transformations.

These requirements allow us to fix Lagrangian, gauge transformations, and operator R^A uniquely for arbitrary $d \geq 4$. For the case of spin 2 conformal field in $d = 4$, we find alternative derivation based on approach of Ref.[38]. This alternative derivation is outlined in Appendix C. In Appendix C, we also present ordinary-derivative form of spin 2 conformal theory in general gravitational background.

¹⁴Note that, as compared to normalization used in (5.29),(5.48) we made re-scaling $\phi_{2,1}^{AB} \rightarrow \frac{1}{2}\phi_{2,1}^{AB}$, $\phi_{2,-1}^{AB} \rightarrow \frac{1}{2}\phi_{2,-1}^{AB}$ in (5.78)-(5.82).

6 Spin 1/2 conformal fermionic Dirac field

We now extend our discussion to the case of conformal half-integer spin fermionic fields. As a warm up we start with fermionic spin 1/2 Dirac field. To make contact with studies in earlier literature we start with presentation of the standard higher-derivative formulation for the fermionic field.

6.1 Higher-derivative formulation of spin 1/2 conformal fermionic field

In the framework of the standard approach, spin 1/2 conformal fermionic field propagating in flat space of arbitrary dimension d is described by Lagrangian

$$i\mathcal{L} = \bar{\psi} \square^k \not{\partial} \psi, \quad (6.1)$$

where k is positive integer, $\bar{\psi} = \psi^\dagger \gamma^0$, and ψ stands for complex-valued non-chiral Dirac field¹⁵. Here and below spinor indices are implicit. For $k = 0$, the Lagrangian (6.1) describes field associated with unitary representation of the conformal algebra $so(d, 2)$. For $k \geq 1$, fermionic field described by Lagrangian associates with non-unitary representation of the conformal algebra. The field ψ has conformal dimension

$$\Delta_\psi = \frac{d-1}{2} - k. \quad (6.2)$$

6.2 Ordinary-derivative formulation of spin 1/2 conformal fermionic field

Field content. In the framework of ordinary-derivative approach, a dynamical system that on-shell equivalent to the single non-chiral Dirac field ψ with Lagrangian (6.1) and conformal dimension (6.2) is described by $2k + 1$ non-chiral Dirac fields¹⁶,

$$\psi_{0,k'}, \quad k' = -k, -k+2, \dots, k-2, k, \quad (6.3)$$

$$\psi_{0,k'}, \quad k' = -k+1, -k+3, \dots, k-3, k-1, \quad (6.4)$$

$$k - \text{arbitrary positive integer}. \quad (6.5)$$

Subscript 0 in $\psi_{0,k'}$ denotes integer part of Lorentz algebra spin, while the subscript k' determines conformal dimensions of the fields $\psi_{0,k'}$ (6.3),(6.4):

$$\Delta_{\psi_{0,k'}} = \frac{d-1}{2} + k'. \quad (6.6)$$

¹⁵ Chiral fermionic field can be introduced by allowing the field ψ to be positive-chirality (or negative-chirality).

¹⁶ Chiral fermionic fields can be introduced by allowing the fields in (6.3) to be positive-chirality (or negative-chirality), while the fields in (6.4) to be negative-chirality (or positive-chirality).

We note that, on-shell, the field $\psi_{0,-k}$ in (6.3) can be identified with the generic fermionic field (6.1),

$$\psi = \psi_{0,-k} . \quad (6.7)$$

In order to obtain the Lagrangian description of a conformal field in an easy-to-use form we use the creation and annihilation operators v^\oplus , v^\ominus , \bar{v}^\ominus , \bar{v}^\oplus . Using creation operators v^\oplus , v^\ominus , we collect fields (6.3), (6.4) into 2-vector ket-vector $|\psi\rangle$ defined by

$$|\psi\rangle = \begin{pmatrix} |\psi^{(0)}\rangle \\ |\psi^{(1)}\rangle \end{pmatrix} , \quad (6.8)$$

$$\begin{aligned} |\psi^{(0)}\rangle &\equiv \sum_{k'} (v^\oplus)^{\frac{k+k'}{2}} (v^\ominus)^{\frac{k-k'}{2}} \psi_{0,k'}(x) |0\rangle , \\ k' &= -k, -k+2, \dots, k-2, k ; \end{aligned} \quad (6.9)$$

$$\begin{aligned} |\psi^{(1)}\rangle &\equiv \sum_{k'} (v^\oplus)^{\frac{k-1+k'}{2}} (v^\ominus)^{\frac{k-1-k'}{2}} \psi_{0,k'}(x) |0\rangle , \\ k' &= -k+1, -k+3, \dots, k-3, k-1 . \end{aligned} \quad (6.10)$$

Definition of the ket-vector $|\psi\rangle$ (6.8) implies the following algebraic constraints

$$(N_v - k)\pi_+ |\psi\rangle = 0 , \quad (6.11)$$

$$(N_v - k + 1)\pi_- |\psi\rangle = 0 . \quad (6.12)$$

The constraints (6.11) and (6.12) tell us that the ket-vectors $|\psi^{(0)}\rangle$ (6.9) and $|\psi^{(1)}\rangle$ (6.9) are the respective degree k and $k-1$ homogeneous polynomials in the oscillators v^\oplus , v^\ominus .

Lagrangian. Lagrangian of the conformal theory under consideration can be written in the form¹⁷

$$i\mathcal{L} = \langle \psi | E | \psi \rangle , \quad (6.13)$$

where operator E is given by

$$E = \not{D} + m_1 , \quad (6.14)$$

$$m_1 \equiv \bar{v}^\ominus \sigma_- + v^\ominus \sigma_+ . \quad (6.15)$$

Realization of conformal boost symmetries. To complete ordinary-derivative description of the spin 1/2 field we provide realization of the conformal algebra symmetries on the space of the ket-vector $|\psi\rangle$. All that is required is to fix operators M^{AB} , Δ and R^A for the case of spin

¹⁷The bra-vector $\langle \psi |$ is defined according the rule $\langle \psi | = (|\psi\rangle)^\dagger \gamma^0$.

1/2 conformal fermionic field and then use these operators in general relations (2.17)-(2.20) which are valid for arbitrary spin fields. For the case of spin 1/2 field the spin matrix of the Lorentz algebra takes the form

$$M^{AB} = \frac{1}{2}\gamma^{AB}. \quad (6.16)$$

Realization of the operator of conformal dimension Δ on space of $|\phi\rangle$ can be read from (6.6),

$$\Delta = \frac{d-1}{2} + \Delta', \quad (6.17)$$

$$\Delta' \equiv N_{v^\oplus} - N_{v^\ominus}. \quad (6.18)$$

Representation of the operator R^A on space of $|\psi\rangle$ is given by

$$R^A = R_0^A + R_1^A + R_E^A, \quad (6.19)$$

$$R_0^A = r_{0,1}^\Gamma \gamma^A, \quad (6.20)$$

$$R_1^A = r_{1,1} \partial^A, \quad (6.21)$$

$$R_E^A = r_{E,1} \gamma^A E, \quad (6.22)$$

$$r_{0,1}^\Gamma = \bar{v}^\oplus \sigma_- - v^\oplus \sigma_+, \quad (6.23)$$

$$r_{1,1} = -2v^\oplus \bar{v}^\oplus, \quad (6.24)$$

$$r_{E,1} = v^\oplus \tilde{r}_{E,1-} \bar{v}^\oplus \pi_- + v^\oplus \tilde{r}_{E,1+} \bar{v}^\oplus \pi_+, \quad (6.25)$$

$$\tilde{r}_{E,1\pm} = \tilde{r}_{E,1\pm}(\Delta'), \quad (6.26)$$

$$\tilde{r}_{E,1\pm}^\dagger = \tilde{r}_{E,1\pm}, \quad (6.27)$$

where E appearing in (6.22) is defined in (6.14) and $\tilde{r}_{E,1\pm}$ (6.26), subject to hermicity condition (6.27), is arbitrary function of the operator Δ' .

This form of the operator R^A is fixed uniquely by requiring that the Lagrangian be invariant with respect to transformations generated by conformal boost operator K^A .

Two remarks are in order.

i) R_0^A and R_1^A parts of the operator R^A are fixed uniquely, while R_E^A part, in view of arbitrary $\tilde{r}_{E,1\pm}$, is still to be arbitrary. Reason for arbitrariness in R_E^A is obvious. Any global transformations of fermionic fields which are realized as differential operators acting on $|\psi\rangle$ are defined by module of terms proportional $x E$, where x is arbitrary operator satisfying the hermitian conjugation condition $x^\dagger = \gamma^0 x \gamma^0$. This condition, in view (6.27), is respected by operator $x = r_{E,1} \gamma^A$ which enters in R_E^A , (6.22). This, the operator R_E^A generates some non-gauge symmetries of the Lagrangian.

ii) Evaluating commutator $[K^A, K^B]$, we obtain $[K^A, K^B] \sim r_E^{AB} E$, where r_E^{AB} is some operator. In other words, $[K^A, K^B]$ is proportional to the operator E . If we impose requirement $[K^A, K^B] = 0$, which amounts to $r_E^{AB} = 0$, then we obtain equations for $\tilde{r}_{E,1\pm}$. Solution to these equations takes the form:

$$\tilde{r}_{E,1+} = \text{const}, \quad \text{for } k = 1; \quad (6.28)$$

$$\tilde{r}_{E,1\pm} = 0, \quad \text{for } k \geq 2. \quad (6.29)$$

Note that, for $k = 1$, $\tilde{r}_{E,1-} = 0$.

7 Spin 3/2 conformal fermionic Dirac field

We now extend our discussion of fermionic fields to the case of spin 3/2 conformal fermionic field. As before to make contact with studies in earlier literature we start with presentation of the standard, i.e. higher-derivative, formulation for the fermionic spin 3/2 Dirac field. In due course, we review our result concerning the counting of on-shell D.o.F for spin 3/2 conformal field.

7.1 Higher-derivative formulation of spin 3/2 conformal fermionic Dirac field

In the framework of the standard approach, spin 3/2 conformal fermionic Dirac field propagating in flat space of arbitrary dimension $d \geq 4$ is described by Lagrangian

$$i\mathcal{L} = \bar{\psi}^A \square^k P_{3/2}^{AB} \not{\partial} \psi^B, \quad k \equiv \frac{d-2}{2}, \quad (7.1)$$

where ψ^A stands for non-chiral vector-spinor complex-valued Dirac field¹⁸. Spinor indices of the fermionic field ψ^A are implicit. The field ψ^A has conformal dimension $\Delta_{\psi^A} = 1/2$. The operator $P_{3/2}^{AB}$ and its basic properties are as follows [1]:¹⁹

$$P_{3/2}^{AB} \equiv \pi^{AB} - \frac{1}{d-1} \pi^{AA'} \pi^{BB'} \gamma^{A'} \gamma^{B'}, \quad (7.2)$$

$$\pi^{AB} \equiv \eta^{AB} - \frac{\partial^A \partial^B}{\square}, \quad (7.3)$$

¹⁸ Chiral spin 3/2 fermionic field can be introduced by allowing the field ψ^A to be positive-chirality (or negative-chirality).

¹⁹ In Ref. [1], operator $P_{3/2}^{AB}$ was given for $d = 4$. Generalization to arbitrary d is straightforward. All that is required is to respect the relations (7.4),(7.5) and the hermicity condition $(i\gamma^0 P_{3/2}^{AB} \not{\partial})^\dagger = i\gamma^0 P_{3/2}^{BA} \not{\partial}$.

$$\partial^A P_{3/2}^{AB} = 0, \quad P_{3/2}^{AB} \partial^B = 0, \quad (7.4)$$

$$\gamma^A P_{3/2}^{AB} = 0, \quad P_{3/2}^{AB} \gamma^B = 0. \quad (7.5)$$

Lagrangian (7.1) is invariant with respect to gauge transformations

$$\delta\psi^A = \partial^A \epsilon, \quad (7.6)$$

$$\delta\psi^A = \gamma^A \lambda, \quad (7.7)$$

where ϵ and λ are parameters of gauge transformations.

We now discuss on-shell D.o.F of the conformal theory under consideration. As before, for this purpose, we use fields transforming in representations of the $so(d-2)$ algebra. Namely, we classify on-shell D.o.F by spin labels of the $so(d-2)$ algebra. One can prove (see Appendix D for details) that on-shell D.o.F are described by $2k+1$ non-chiral vector-spinor fields $\psi_{1,k'}^I$ and $2k-1$ non-chiral spinor fields $\psi_{0,k'}$ ²⁰:

$$\psi_{1,k'}^I, \quad k' = -k, -k+2, \dots, k-2, k; \quad (7.8)$$

$$\psi_{1,k'}^I, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (7.9)$$

$$\psi_{0,k'}, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (7.10)$$

$$\psi_{0,k'}, \quad k' = -k+2, -k+4, \dots, k-4, k-2. \quad (7.11)$$

The $\psi_{1,k'}^I, \psi_{0,k'}$ are *complex-valued* fermionic fields and they transform in the respective non-chiral vector-spinor and non-chiral spinor representations of the $so(d-2)$ algebra²¹. We note that on-shell D.o.F given in (7.11) appear only for $d \geq 6$ (i.e. $k \geq 2$).

Total number of *complex-valued* on-shell D.o.F shown in (7.8)-(7.11) is given by:

$$\nu = 2^{\frac{d-2}{2}} d(d-3). \quad (7.12)$$

We note that this ν is a sum of D.o.F for fields given in (7.8)-(7.11)²²:

$$\nu = \sum_{k'} \nu(\psi_{1,k'}^I) + \sum_{k'} \nu(\psi_{1,k'}^I) + \sum_{k'} \nu(\psi_{0,k'}) + \sum_{k'} \nu(\psi_{0,k'}), \quad (7.13)$$

²⁰ If the generic conformal field ψ has positive (or negative) chirality, then the fields in (7.8),(7.10) should have positive (or negative) chirality, while the fields in (7.9),(7.11) should have negative (or positive) chirality.

²¹ These fields are subject to the standard constraints $\gamma^I \psi_{0,k'}^I = 0, \Pi^{\hat{+}} \psi_{0,k'}^I = 0, \Pi^{\hat{-}} \psi_{0,k'} = 0$ (see Appendix D for details).

²² Total D.o.F ν (7.12) for spin 3/2 conformal fermionic field in $d=4$ was found in [1]. Decomposition of ν (7.13) into irreps of the $so(d-2)$ algebra for the case of $d=4$ spin 3/2 conformal field was carried out in [37]. In Appendix D, we use light-cone approach to generalize these results to the case of arbitrary $d \geq 4$.

$$\nu(\psi_{1,k'}^I) = 2^{\frac{d-2}{2}}(d-3), \quad k' = -k, -k+2, \dots, k-2, k; \quad (7.14)$$

$$\nu(\psi_{1,k'}^I) = 2^{\frac{d-2}{2}}(d-3), \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (7.15)$$

$$\nu(\psi_{0,k'}) = 2^{\frac{d-2}{2}}, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (7.16)$$

$$\nu(\psi_{0,k'}) = 2^{\frac{d-2}{2}}, \quad k' = -k+2, -k+4, \dots, k-4, k-2. \quad (7.17)$$

7.2 Ordinary-derivative formulation of spin 3/2 conformal fermionic Dirac field

Field content. To discuss ordinary-derivative and gauge invariant formulation of spin 3/2 conformal non-chiral Dirac field in flat space of dimension $d \geq 4$ we use $2k+1$ non-chiral vector-spinor Dirac fields $\psi_{1,k'}^A$ and $2k-1$ non-chiral spinor Dirac fields $\psi_{0,k'}$:

$$\psi_{1,k'}^A, \quad k' = -k, -k+2, \dots, k-2, k; \quad (7.18)$$

$$\psi_{1,k'}^A, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (7.19)$$

$$\psi_{0,k'}, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (7.20)$$

$$\psi_{0,k'}, \quad k' = -k+2, -k+4, \dots, k-4, k-2; \quad (7.21)$$

$$k \equiv \frac{d-2}{2}, \quad (7.22)$$

where spinor indices of the fermionic fields $\psi_{1,k'}^A$ and $\psi_{0,k'}$ are implicit. The fields $\psi_{1,k'}^A$ and $\psi_{0,k'}$ are the respective non-chiral vector-spinor fields and non-chiral spinor fields of the Lorentz algebra $so(d-1, 1)^{23}$. We note that fields in (7.21) appear only for $d \geq 6$ (i.e. $k \geq 2$). Also, we note that fields in (7.18)-(7.21) have conformal dimensions

$$\Delta_{\psi_{1,k'}^A} = \frac{d-1}{2} + k', \quad \Delta_{\psi_{0,k'}} = \frac{d-1}{2} + k'. \quad (7.23)$$

In order to obtain the gauge invariant description in an easy-to-use form we use, as usually, a set of the oscillators $\alpha^A, \zeta, v^\oplus, v^\ominus$. The fields (7.18)-(7.21) can then be collected into a 2-vector ket-vector $|\psi\rangle$ defined by

$$|\psi\rangle = |\psi_1\rangle + \zeta|\psi_0\rangle, \quad (7.24)$$

²³ Chiral fermionic fields can be introduced by allowing the fields (7.18), (7.20) in to be positive-chirality (or negative-chirality), while the fields (7.19), (7.21) to be negative-chirality (or positive-chirality).

where 2-vector ket-vectors $|\psi_1\rangle, |\psi_0\rangle$ are defined by

$$|\psi_1\rangle = \begin{pmatrix} |\psi_1^{(0)}\rangle \\ |\psi_1^{(1)}\rangle \end{pmatrix}, \quad |\psi_0\rangle = \begin{pmatrix} |\psi_0^{(0)}\rangle \\ |\psi_0^{(1)}\rangle \end{pmatrix}, \quad (7.25)$$

$$|\psi_1^{(0)}\rangle \equiv \sum_{k'} \alpha^A(v^\oplus)^{\frac{k+k'}{2}} (v^\ominus)^{\frac{k-k'}{2}} \psi_{1,k'}^A(x) |0\rangle, \\ k' = -k, -k+2, \dots, k-2, k; \quad (7.26)$$

$$|\psi_1^{(1)}\rangle \equiv \sum_{k'} \alpha^A(v^\oplus)^{\frac{k-1+k'}{2}} (v^\ominus)^{\frac{k-1-k'}{2}} \psi_{1,k'}^A(x) |0\rangle, \\ k' = -k+1, -k+3, \dots, k-3, k-1; \quad (7.27)$$

$$|\psi_0^{(0)}\rangle \equiv \sum_{k'} (v^\oplus)^{\frac{k-1+k'}{2}} (v^\ominus)^{\frac{k-1-k'}{2}} \psi_{0,k'}(x) |0\rangle, \\ k' = -k+1, -k+3, \dots, k-3, k-1; \quad (7.28)$$

$$|\psi_0^{(1)}\rangle \equiv \sum_{k'} (v^\oplus)^{\frac{k-2+k'}{2}} (v^\ominus)^{\frac{k-2-k'}{2}} \psi_{0,k'}(x) |0\rangle, \\ k' = -k+2, -k+4, \dots, k-4, k-2. \quad (7.29)$$

It is easy to see that the ket-vector $|\psi\rangle$ (7.24) satisfies the following algebraic constraints:

$$(N_\alpha + N_\zeta - 1)|\psi\rangle = 0, \quad (7.30)$$

$$(N_\zeta + N_v - k)\pi_+|\psi\rangle = 0, \quad (7.31)$$

$$(N_\zeta + N_v - k + 1)\pi_-|\psi\rangle = 0. \quad (7.32)$$

The constraint (7.30) tell us that the ket-vector $|\psi\rangle$ (7.24) is degree 1 homogeneous polynomial in the oscillators α^A, ζ . From the constraints (7.31),(7.32), we learn that

- a) the ket-vector $|\psi_1^{(0)}\rangle$ (7.26) is degree k homogeneous polynomial in v^\oplus, v^\ominus ;
- b) the ket-vectors $|\psi_1^{(1)}\rangle$ (7.27), $|\psi_0^{(0)}\rangle$ (7.28) are degree $k-1$ homogeneous polynomials in v^\oplus, v^\ominus ;
- c) the ket-vector $|\psi_0^{(1)}\rangle$ (7.29) is degree $k-2$ homogeneous polynomial in v^\oplus, v^\ominus .

We note that the ket-vector $|\psi_0^{(1)}\rangle$ appear only in $d \geq 6$ (i.e. $k \geq 2$).

Lagrangian. Lagrangian we found takes the form

$$i\mathcal{L} = \langle \psi | E | \psi \rangle , \quad (7.33)$$

where operator E is given by

$$E = E_{(1)} + E_{(0)} , \quad (7.34)$$

$$E_{(1)} \equiv \not{\partial} - \alpha \partial \gamma \bar{\alpha} - \gamma \alpha \bar{\alpha} \partial + \gamma \alpha \not{\partial} \gamma \bar{\alpha} , \quad (7.35)$$

$$E_{(0)} = (1 - \gamma \alpha \gamma \bar{\alpha}) m_1 + \gamma \alpha m_4 - m_4^\dagger \gamma \bar{\alpha} , \quad (7.36)$$

$$m_1 = \frac{d}{d - 2N_\zeta} (\bar{v}^\ominus \sigma_- + v^\ominus \sigma_+) , \quad (7.37)$$

$$m_4 = \left(\frac{d-1}{d-2} \right)^{1/2} v^\ominus \bar{\zeta} , \quad (7.38)$$

$$m_4^\dagger = \left(\frac{d-1}{d-2} \right)^{1/2} \zeta \bar{v}^\ominus . \quad (7.39)$$

We note that $E_{(1)}$ (7.35) is standard first-order Rarita-Schwinger operator rewritten in terms of the oscillators.

Gauge transformations. We now discuss gauge symmetries of the Lagrangian. To this end we introduce gauge transformation parameters,

$$\epsilon_{0,k'-1} , \quad k' = -k, -k+2, \dots, k-2, k ; \quad (7.40)$$

$$\epsilon_{0,k'-1} \quad k' = -k+1, -k+3, \dots, k-3, k-1 . \quad (7.41)$$

The gauge transformations parameters $\epsilon_{0,k'}$ are non-chiral spinor fields of the Lorentz algebra $so(d-1, 1)$. As usually, we then collect gauge transformation parameters in ket-vector $|\epsilon\rangle$ defined by

$$\epsilon = \begin{pmatrix} |\epsilon^{(0)}\rangle \\ |\epsilon^{(1)}\rangle \end{pmatrix} , \quad (7.42)$$

$$\begin{aligned} |\epsilon^{(0)}\rangle &\equiv \sum_{k'} (v^\oplus)^{\frac{k+k'}{2}} (v^\ominus)^{\frac{k-k'}{2}} \epsilon_{0,k'-1}(x) |0\rangle , \\ k' &= -k, -k+2, \dots, k-2, k ; \end{aligned} \quad (7.43)$$

$$\begin{aligned} |\epsilon^{(1)}\rangle &\equiv \sum_{k'} (v^\oplus)^{\frac{k-1+k'}{2}} (v^\ominus)^{\frac{k-1-k'}{2}} \epsilon_{0,k'-1}(x) |0\rangle , \\ k' &= -k+1, -k+3, \dots, k-3, k-1 . \end{aligned} \quad (7.44)$$

The ket-vector of gauge transformation parameters $|\epsilon\rangle$ satisfies the algebraic constraints

$$(N_v - k)\pi_+|\epsilon\rangle = 0, \quad (7.45)$$

$$(N_v - k + 1)\pi_-|\epsilon\rangle = 0, \quad (7.46)$$

which tell us that the ket-vector $|\epsilon^{(0)}\rangle$ (7.43) is degree k homogeneous polynomial in v^\oplus, v^\ominus , while the ket-vector $|\epsilon^{(1)}\rangle$ (7.44) is degree $k - 1$ homogeneous polynomial in v^\oplus, v^\ominus .

Gauge transformations take the form

$$\delta|\psi\rangle = (\alpha\partial + \mathcal{F})|\epsilon\rangle, \quad (7.47)$$

$$\mathcal{F} \equiv f_1 + \gamma\alpha f_2, \quad (7.48)$$

$$f_1 = \left(\frac{d-1}{d-2}\right)^{1/2} \zeta \bar{v}^\ominus, \quad (7.49)$$

$$f_2 = \frac{1}{d-2} (\bar{v}^\ominus \sigma_- + v^\ominus \sigma_+). \quad (7.50)$$

Realization of conformal boost symmetries. To complete ordinary-derivative description of the spin 3/2 field we provide realization of the conformal algebra symmetries on the space of the ket-vector $|\psi\rangle$. All that is required is to fix operators M^{AB} , Δ and R^A for the case of spin 3/2 conformal fermionic field and then use these operators in (2.17)-(2.20). For the case of spin 3/2 field, the spin matrix of the Lorentz algebra takes the form

$$M^{AB} = \alpha^A \bar{\alpha}^B - \alpha^B \bar{\alpha}^A + \frac{1}{2} \gamma^{AB}. \quad (7.51)$$

Realization of the operator of conformal dimension Δ on space of $|\psi\rangle$ can be read from (7.23),

$$\Delta = \frac{d-1}{2} + \Delta', \quad (7.52)$$

$$\Delta' \equiv N_{v^\oplus} - N_{v^\ominus}. \quad (7.53)$$

Representation of the operator R^A on space of $|\psi\rangle$ is given by

$$R^A = R_0^A + R_1^A + R_G^A + R_E^A, \quad (7.54)$$

$$\begin{aligned} R_0^A &= r_{0,1}^\Gamma (\gamma^A - \frac{2}{d-2} \gamma \alpha \bar{\alpha}^A) \\ &\quad + r_{0,1} (\alpha^A - \frac{1}{d-2} \gamma \alpha \gamma^A) - r_{0,1}^\dagger \bar{\alpha}^A, \end{aligned} \quad (7.55)$$

$$R_1^A = r_{1,1} \partial^A, \quad (7.56)$$

$$R_G^A = G r_G^A, \quad (7.57)$$

$$R_E^A = r_E^A E, \quad (7.58)$$

$$r_{0,1}^\Gamma = \frac{d}{d-2N_\zeta} (\bar{v}^\oplus \sigma_- - v^\oplus \sigma_+), \quad (7.59)$$

$$r_{0,1} = 2 \left(\frac{d-1}{d-2} \right)^{1/2} v^\oplus \bar{\zeta}, \quad (7.60)$$

$$r_{0,1}^\dagger = 2 \left(\frac{d-1}{d-2} \right)^{1/2} \zeta \bar{v}^\oplus, \quad (7.61)$$

$$r_{1,1} = -2v^\oplus \bar{v}^\oplus, \quad (7.62)$$

$$r_G^A = r_{G,1} \bar{\alpha}^A + r_{G,2} \gamma^A + r_{G,3} \gamma^A \gamma \bar{\alpha}, \quad (7.63)$$

$$\begin{aligned} r_E^A &= r_{E,1} \gamma^A + r_{E,2} \alpha^A \gamma \bar{\alpha} + r_{E,3} \gamma \alpha \bar{\alpha}^A + r_{E,4} \gamma \alpha \gamma^A \gamma \bar{\alpha} \\ &\quad + r_{E,5} \alpha^A + r_{E,6} \bar{\alpha}^A + r_{E,7} \gamma \alpha \gamma^A + r_{E,8} \gamma^A \gamma \bar{\alpha} + r_{E,9} \gamma^A, \end{aligned} \quad (7.64)$$

$$r_{G,a} = v^\oplus \tilde{r}_{G,a-} \bar{v}^\oplus \pi_- + v^\oplus \tilde{r}_{G,a+} \bar{v}^\oplus \pi_+, \quad a = 1, 3; \quad (7.65)$$

$$r_{G,a} = v^\oplus \tilde{r}_{G,a-} \bar{v}^\oplus \bar{\zeta} \sigma_- + v^\oplus v^\oplus \tilde{r}_{G,a+} \bar{\zeta} \sigma_+, \quad a = 2; \quad (7.66)$$

$$r_{E,a} = (v^\oplus \tilde{r}_{E,a-} \bar{v}^\oplus \pi_- + v^\oplus \tilde{r}_{E,a+} \bar{v}^\oplus \pi_+) (1 - N_\zeta), \quad a = 1; \quad (7.67)$$

$$r_{E,a} = v^\oplus \tilde{r}_{E,a-} \bar{v}^\oplus \pi_- + v^\oplus \tilde{r}_{E,a+} \bar{v}^\oplus \pi_+, \quad a = 2, 3, 4; \quad (7.68)$$

$$r_{E,a} = v^\oplus \tilde{r}_{E,a-} \bar{v}^\oplus \bar{\zeta} \sigma_- + v^\oplus v^\oplus \tilde{r}_{E,a+} \bar{\zeta} \sigma_+, \quad a = 5, 7; \quad (7.69)$$

$$r_{E,a} = \zeta \tilde{r}_{E,a-} \bar{v}^\oplus \bar{v}^\oplus \sigma_- + \zeta v^\oplus \tilde{r}_{E,a+} \bar{v}^\oplus \sigma_+, \quad a = 6, 8; \quad (7.70)$$

$$r_{E,a} = (v^\oplus \tilde{r}_{E,a-} \bar{v}^\oplus \pi_- + v^\oplus \tilde{r}_{E,a+} \bar{v}^\oplus \pi_+) N_\zeta, \quad a = 9; \quad (7.71)$$

$$\tilde{r}_{G,a\pm} = \tilde{r}_{G,a\pm}(\Delta'), \quad a = 1, 2, 3. \quad (7.72)$$

$$\tilde{r}_{E,a\pm} = \tilde{r}_{E,a\pm}(\Delta'), \quad a = 1, \dots, 9; \quad (7.73)$$

$$\tilde{r}_{E,a\pm}^\dagger = \tilde{r}_{E,a\pm} \quad a = 1, 4, 9; \quad (7.74)$$

$$\tilde{r}_{E,2\pm}^\dagger = \tilde{r}_{E,3\pm}, \quad \tilde{r}_{E,5\pm}^\dagger = -r_{E,6\mp}, \quad \tilde{r}_{E,7\pm}^\dagger = -\tilde{r}_{E,8\mp}, \quad (7.75)$$

where G (7.57) stands for operator of gauge transformation (7.47), $G = \alpha \partial + \mathcal{F}$, while E appearing in (7.58) is defined in (7.34), and $\tilde{r}_{G,a\pm}, \tilde{r}_{E,a\pm}$ (7.72), (7.73) are arbitrary functions of the operator Δ' . The functions $\tilde{r}_{E,a\pm}$ are subject to hermitian conjugation rules in (7.74), (7.75). Note that the operators $r_{G,2}$ and $r_{E,a}$, $a = 5, 6, 7, 8$ are non-trivial only for $k \geq 2$ (i.e. $d \geq 6$).

This form of the operator R^A is fixed uniquely by requiring that the Lagrangian be invariant with respect to transformations generated by the conformal boost operator K^A .

Two remarks are in order.

i) R_0^A and R_1^A parts of the operator R^A are fixed uniquely, while R_G^A and R_E^A parts, in view of arbitrary $\tilde{r}_{G,a\pm}, \tilde{r}_{E,a\pm}$, are still to be arbitrary. Reason for arbitrariness in R_G^A and R_E^A is obvious. Arbitrariness in R_G^A is because of global transformations of fermionic spin 3/2 gauge field are defined by module of gauge transformations. Reason for arbitrariness in R_E^A is the same as for the case of spin 1/2 field.

ii) Evaluating commutator $[K^A, K^B]$, we obtain $[K^A, K^B] \sim Gr_G^{AB} + r_E^{AB} E$, where r_G^{AB} and r_E^{AB} are some differential operators. If we impose requirement $[K^A, K^B] = 0$, then we

obtain equations for $\tilde{r}_{G,a\pm}$ and $\tilde{r}_{E,a\pm}$. Because explicit form of solution of those equations is not illuminating we do not present it here.

Note that the simplest representation for R^A is achieved by taking $R_G^A = 0$, $R_E^A = 0$.

8 Conclusions

We have developed the ordinary-derivative formulation of conformal fields in flat space of arbitrary dimension. In this paper we applied this formalism to the study of low spin fields. Because the formulation we presented is based on use of oscillator realization of spin degrees of freedom of gauge fields it allows straightforward generalization to higher-spin conformal fields. Comparison of formulation we developed with other approaches available in the literature leads us to the conclusion that this is a very interesting and attractive formulation.

The results presented here should have a number of interesting applications and generalizations, some of which are:

i) generalization to supersymmetric conformal field theories and applications to conformal supergravities in various dimensions [39]-[44]. Note that in this paper we work out Lagrangians and realization of conformal symmetries for all fields that appear in supermultiplets of conformal supergravities and involve higher-derivative contributions to Lagrangian of conformal supergravity. The first step in this direction would be understanding of how the supersymmetries are realized in the framework of our approach which suggests new gauge fields content for study of (super)conformal gravities.

ii) extension of our approach to interacting (super)conformal low spin²⁴ and higher-spin field theories. Such theories taken to be in cubic approximation were discussed [46, 47]. Our approach to conformal theories is based on new realization of conformal gauge symmetries via Stueckelberg fields. In our approach, use of Stueckelberg fields is very similar to the one in gauge invariant formulation of massive fields. Stueckelberg fields provides interesting possibilities for study of interacting massive gauge fields (see e.g. [48, 49]). So we think that application of our approach to the (super)conformal interacting fields²⁵ should lead to new interesting development.

iii) BRST approach is one of powerful approaches to analysis of various aspects of relativistic dynamics (see e.g. [52]). Though BRST approach was extended for study of higher-derivative theories (see e.g. [53]), it seems that this approach is conveniently adopted for ordinary-derivative formulation. BRST approach was extensively developed in recent time

²⁴ In the framework higher-derivative formulation, uniqueness of interacting spin 2 conformal field theory was discussed in [45].

²⁵ Recent discussion of massive supermultiplets in the framework gauge invariant (Stueckelberg) approach may be found in [50, 51].

(see e.g. [54]-[59]) and was applied to ordinary-derivative Lagrangian theories of massless and massive fields in flat and AdS space. Because AdS theories and conformal theories share many algebraic properties, application of previously developed various BRST methods to study of ordinary-derivative conformal field theories should be relatively straightforward.

iv) There are other various interesting approaches in the literature which could be used to discuss ordinary-derivative formulation of conformal theories. This is to say that various formulations in terms of unconstrained fields in flat space and (A)dS space may be found e.g. in [60]-[62].

v) Mixed symmetry fields [63, 64] have attracted considerable interest in recent time (see e.g. [65]-[70]). In AdS space, massless mixed symmetry fields, in contrast to massless fields in Minkowski space whose physical degrees of freedom transform in irreps of $o(d-2)$ algebra, reduce to a number of irreps of $so(d-2)$ algebra. In other words, not every massless field in flat space admits a deformation to AdS_d with the same number of degrees of freedom [71]. It would be interesting to check this phenomena for the case of mixed symmetry conformal fields.

vi) extension of our approach to light-cone gauge conformal fields and application to analysis of interaction of conformal fields to composite operators constructed out fields of supersymmetric YM theory²⁶. We expect that a quantization of the Green-Schwarz AdS superstring with a Ramond - Ramond charge will be available only in the light-cone gauge [72, 73]. Therefore it seems that from the stringy perspective of AdS/CFT correspondence the light-cone approach to conformal field theory is the fruitful direction to go.

We strongly believe that the approach developed in this paper will be useful for better understanding conformal field theory.

Acknowledgments. This work was supported by the INTAS project 03-51-6346, by the RFBR Grant No.05-02-17217, RFBR Grant for Leading Scientific Schools, Grant No. 1578-2003-2 and Russian Science Support Foundation.

Appendix A Counting of on-shell D.o.F for spin 1 conformal field

We analyze on-shell D.o.F for spin 1 conformal field that is described by Lagrangian (4.1). To this end we use framework of light-cone gauge approach which turns out to be helpful for our purpose²⁷. Lagrangian (4.1) leads to the following E.o.M:

$$\square^{1+k}\phi^A - \partial^A \square^k \partial \phi = 0, \quad (\text{A.1})$$

²⁶ Approach developed in [74] should streamline such a analysis

²⁷ Discussion of alternative method for counting of on-shell D.o.F may be found in [37, 34].

where here and below we use the notation $\partial\phi \equiv \partial^A\phi^A$. Making use of light-cone gauge²⁸

$$\phi^+ = 0, \quad (\text{A.2})$$

we obtain from ‘+’ components of Eq.(A.1),

$$\partial^+\square^k\partial\phi = 0. \quad (\text{A.3})$$

As usually, kernel of operator ∂^+ is assumed to be trivial. Therefore Eq.(A.3) amounts to equation $\square^k\partial\phi = 0$. Plugging the latter equation in (A.1) gives $\square^{1+k}\phi^A = 0$. To summarize, in light-cone gauge, the generic Eqs.(A.1) amount to equations

$$\square^{1+k}\phi^A = 0, \quad (\text{A.4})$$

$$\square^k\partial\phi = 0. \quad (\text{A.5})$$

We now transform Eqs.(A.4),(A.5) into ordinary-derivative form. Introducing $k + 1$ vector fields

$$\phi_{1,k'}^A, \quad k' = -k, -k + 2, \dots, k - 2, k, \quad (\text{A.6})$$

and k scalar fields

$$\phi_{0,k'}, \quad k' = -k + 1, -k + 3, \dots, k - 3, k - 1, \quad (\text{A.7})$$

where the generic conformal vector field ϕ^A is identified as

$$\phi_{1,-k}^A \equiv \phi^A, \quad (\text{A.8})$$

we derive the ordinary-derivative form of Eqs.(A.4), (A.5)

$$\square\phi_{1,k'}^A - \phi_{1,k'+2}^A = 0, \quad k' = -k, -k + 2, \dots, k - 2, k; \quad (\text{A.9})$$

$$\partial\phi_{1,k'} - \phi_{0,k'+1} = 0, \quad k' = -k, -k + 2, \dots, k - 2, k; \quad (\text{A.10})$$

$$\square\phi_{0,k'} - \phi_{0,k'+2} = 0, \quad k' = -k + 1, -k + 3, \dots, k - 3, k - 1; \quad (\text{A.11})$$

where we use the conventions $\partial\phi_{1,k'} \equiv \partial^A\phi_{1,k'}^A$,

$$\phi_{1,k+2}^A \equiv 0, \quad \phi_{0,k+1} \equiv 0. \quad (\text{A.12})$$

Note that light-cone gauge for generic field (A.2) implies the following light-cone gauge for fields in (A.6),

$$\phi_{1,k'}^+ = 0, \quad k' = -k, -k + 2, \dots, k - 2, k. \quad (\text{A.13})$$

²⁸ Light-cone coordinates in \pm directions are defined as $x^\pm = (x^{d-1} \pm x^0)/\sqrt{2}$ and x^+ is taken to be a light-cone time. We adopt the conventions: $\partial^I = \partial_I \equiv \partial/\partial x^I$, $\partial^\pm = \partial_\mp \equiv \partial/\partial x^\mp$, $I, J = 1, \dots, d - 2$. Lorentz algebra vector X^A is decomposed as $X^A = (X^+, X^-, X^I)$. Note that $X^+ = X_-$, $X^- = X_+$, $X^I = X_I$.

Light-cone gauge (A.13) and constrains (A.10) allow us to express $\phi_{1,k'}^-$ in terms of on-shell vector fields $\phi_{1,k}^I$ and scalar fields $\phi_{0,k'}$,

$$\phi_{1,k'}^- = -\frac{\partial^I}{\partial^+} \phi_{1,k'}^I + \frac{1}{\partial^+} \phi_{0,k'+1}. \quad (\text{A.14})$$

To summarize, we have $k+1$ on-shell vector fields $\phi_{1,k'}^I$, $k' = -k, -k+2, \dots, k-2, k$ and k on-shell scalar fields $\phi_{0,k'}$, $k' = -k+1, -k+3, \dots, k-3, k-1$.

Appendix B Counting of on-shell D.o.F for spin 2 conformal field

We analyze on-shell D.o.F for spin 2 conformal field that is described by Lagrangian (5.3). Lagrangian (5.3) leads to E.o.M

$$\square^{k+1} P_{A'B'}^{AB} \phi^{A'B'} = 0, \quad (\text{B.1})$$

which are invariant with respect to linearized diffeomorphism gauge symmetries and Weyl conformal gauge symmetries given in (5.6) and (5.7) respectively. We analyze E.o.M (B.1) by using light-cone gauge to fix diffeomorphism gauge symmetries and traceless condition to fix Weyl gauge symmetry,

$$\phi^{+A} = 0, \quad (\text{B.2})$$

$$\phi^{AA} = 0. \quad (\text{B.3})$$

Using (B.2),(B.3), we find that ‘++’ component of E.o.M (B.1),

$$\square^{k+1} P_{A'B'}^{++} \phi^{A'B'} = 0, \quad (\text{B.4})$$

amounts to the constraint

$$\square^{k-1} \partial \partial \phi = 0, \quad (\text{B.5})$$

where here and below we use the notation $\partial \partial \phi \equiv \partial^A \partial^B \phi^{AB}$. Plugging (B.5) in (B.1) we obtain

$$\square^{k+1} \phi^{AB} - \partial^A \square^k (\partial \phi)^B - \partial^B \square^k (\partial \phi)^A = 0, \quad (\text{B.6})$$

where here and below we use the notation $(\partial \phi)^A \equiv \partial^B \phi^{AB}$. ‘A+’ components of Eqs.(B.6) give

$$\square^k (\partial \phi)^A = 0. \quad (\text{B.7})$$

To summarize, the generic E.o.M (B.1) amount to equations

$$\square^{k+1}\phi^{AB} = 0, \quad (\text{B.8})$$

$$\square^k(\partial\phi)^A = 0, \quad (\text{B.9})$$

$$\square^{k-1}\partial\partial\phi = 0. \quad (\text{B.10})$$

Making use of the notation

$$\phi_{2,-k}^{AB} \equiv \phi^{AB}, \quad (\text{B.11})$$

we rewrite Eqs.(B.8)-(B.10) into ordinary-derivative form

$$\square\phi_{2,k'}^{AB} - \phi_{2,k'+2}^{AB} = 0, \quad k' = -k, -k+2, \dots, k-2, k; \quad (\text{B.12})$$

$$\square\phi_{1,k'}^A - \phi_{1,k'+2}^A = 0, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (\text{B.13})$$

$$\square\phi_{0,k'} - \phi_{0,k'+2} = 0, \quad k' = -k+2, -k+4, \dots, k-4, k-2; \quad (\text{B.14})$$

$$(\partial\phi_{2,k'})^A - \phi_{1,k'+1}^A = 0, \quad k' = -k, -k+2, \dots, k-2, k; \quad (\text{B.15})$$

$$\partial\phi_{1,k'} - \phi_{0,k'+1} = 0, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (\text{B.16})$$

where we use the conventions $(\partial\phi_{2,k'})^A \equiv \partial^B\phi_{2,k'}^{AB}$, $\partial\phi_{1,k'} \equiv \partial^A\phi_{1,k'}^A$,

$$\phi_{2,k+2}^{AB} \equiv 0, \quad \phi_{1,k+1}^A \equiv 0, \quad \phi_{0,k} \equiv 0. \quad (\text{B.17})$$

Gauge conditions (B.2),(B.3) imply

$$\phi_{2,k'}^{+A} = 0, \quad \phi_{2,k'}^{II} = 0, \quad k' = -k, -k+2, \dots, k-2, k; \quad (\text{B.18})$$

$$\phi_{1,k'}^+ = 0, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (\text{B.19})$$

Light-cone gauge and constrains (B.15),(B.16) allow us to express non-dynamical fields $\phi_{2,k'}^{-I}$, $\phi_{2,k'}^{-\bar{I}}$, and $\phi_{1,k'}^{-I}$ in terms of dynamical fields $\phi_{2,k'}^{IJ}$, $\phi_{1,k'}^I$, and $\phi_{0,k'}$,

$$\phi_{2,k'}^{-I} = -\frac{\partial^J}{\partial^+}\phi_{2,k'}^{IJ} + \frac{1}{\partial^+}\phi_{1,k'+1}^I. \quad (\text{B.20})$$

$$\phi_{2,k'}^{-\bar{I}} = \frac{\partial^I\partial^J}{\partial^+\partial^+}\phi_{2,k'}^{IJ} - 2\frac{\partial^I}{\partial^+\partial^+}\phi_{1,k'+1}^I + \frac{1}{\partial^+\partial^+}\phi_{0,k'+2}, \quad (\text{B.21})$$

$$\phi_{1,k'}^{-I} = -\frac{\partial^I}{\partial^+}\phi_{1,k'}^I + \frac{1}{\partial^+}\phi_{0,k'+1}. \quad (\text{B.22})$$

To summarize, we have $k+1$ on-shell traceless rank-2 tensor fields $\phi_{1,k'}^{IJ}$, $k' = -k, -k+2, \dots, k-2, k$, k on-shell vector fields $\phi_{1,k'}^I$, $k' = -k+1, -k+3, \dots, k-3, k-1$, and $k-1$ on-shell scalar fields $\phi_{0,k'}$, $k' = -k+2, -k+4, \dots, k-4, k-2$.

Appendix C Derivation of ordinary-derivative Lagrangian for spin 2 conformal field in $d = 4$

In this Appendix, we outline derivation of ordinary-derivative Lagrangian for spin 2 conformal field in $d = 4$ by using the gauge invariant formulation of $4d$ conformal gravity given in Ref.[38]. Thus, we start with Lagrangian in Ref.[38],

$$\mathcal{L} = \frac{1}{8}G_B, \quad -4G_B \equiv \epsilon_{ABCD}\epsilon^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu}^{AB}\mathcal{R}_{\rho\sigma}^{CD}, \quad (\text{C.1})$$

where Gauss-Bonnet G_B , which can conveniently be represented as

$$G_B = \mathcal{R}_{\mu\nu}^{AB}\mathcal{R}_{AB}^{\mu\nu} - 4\mathcal{R}_\mu^A\mathcal{R}_A^\mu + \mathcal{R}^2, \quad \mathcal{R}_\mu^A \equiv \mathcal{R}_{\mu\nu}^{AB}\delta_B^\nu, \quad \mathcal{R} \equiv \mathcal{R}_\mu^A\delta_A^\mu, \quad (\text{C.2})$$

is expressed in terms of generalized Riemann tensor $\mathcal{R}_{\mu\nu}^{AB}$. This tensor, expanded over flat space, takes the form

$$\mathcal{R}_{\mu\nu}^{AB} = \widehat{R}_{\mu\nu}^{AB} + 2(\delta_\mu^A f_\nu^B + 3 \text{ terms}), \quad (\text{C.3})$$

$$\widehat{R}_{\mu\nu}^{AB} = \partial_\mu \widehat{\omega}_\nu^{AB} - \partial_\nu \widehat{\omega}_\mu^{AB}, \quad (\text{C.4})$$

$$\widehat{\omega}_\mu^{AB} = \omega_\mu^{AB}(h) - \delta_\mu^A b^B + \delta_\mu^B b^A, \quad (\text{C.5})$$

$$\widehat{\omega}_\mu^{AB}(h) = \frac{1}{2}(-\partial^A h_\mu^B + \partial^B h_\mu^A). \quad (\text{C.6})$$

The conformal graviton field h_μ^A and fields f_μ^A and b^A appearing in Lagrangian are gauge fields. Their gauge transformations are given by

$$\delta h_\mu^A = \partial_\mu \epsilon^A + \partial^A \epsilon_\mu + 2\delta_\mu^A \epsilon_{(D)}, \quad (\text{C.7})$$

$$\delta f_\mu^A = \partial_\mu \epsilon_{(K)}^A, \quad (\text{C.8})$$

$$\delta b^A = \partial^A \epsilon_{(D)} - 2\epsilon_{(K)}^A, \quad (\text{C.9})$$

where ϵ^A , $\epsilon_{(K)}$, $\epsilon_{(D)}$ are gauge transformation parameters. Because of the generalized Riemann tensor $\mathcal{R}_{\mu\nu}^{AB}$ (C.3) is invariant with respect to gauge transformations (C.7)-(C.9), the Lagrangian (C.1) is also gauge invariant.

We rewrite the Lagrangian (C.1) explicitly in terms of $f_\mu^\nu \equiv f_\mu^\nu \delta_A^\nu$ and $b^\mu \equiv b^A \delta_A^\mu$,

$$\mathcal{L} = -2R_\mu^\nu f_\nu^\mu + Rf - 4f_\mu^\nu \partial_\nu b^\mu + 4f\partial b - 4f_\mu^\nu f_\nu^\mu + 4f^2, \quad (\text{C.10})$$

where $f \equiv f_\mu^\mu$, and $R_\mu^\nu \equiv R_\mu^A \delta_A^\nu$ is linearized Ricci tensor of conformal graviton field $h_\mu^\nu \equiv h_\mu^A \delta_A^\nu$,

$$R_{\mu\nu} = \frac{1}{2}(-\square h_{\mu\nu} + \partial_\mu(\partial h)_\nu + \partial_\nu(\partial h)_\mu - \partial_\mu\partial_\nu h), \quad (\text{C.11})$$

$R = R_\mu^\mu$, $(\partial h)_\mu \equiv \partial_\nu h_\mu^\nu$, $h = h_\mu^\mu$, $h_{\mu\nu} = h_\mu^\rho \eta_{\rho\nu}$, and $\eta_{\mu\nu}$ is a flat metric tensor.

Before deriving of ordinary-derivative Lagrangian we recall how the standard higher-derivative Lagrangian is obtained from Lagrangian (C.10). To this end we consider E.o.M for the field f_μ^ν ,

$$\partial_{f_\mu^\nu} \mathcal{L} = 0, \quad (\text{C.12})$$

which allows us to solve the field f_μ^ν in terms of the fields h_μ^ν and b^μ ,

$$f_\mu^\nu = -\frac{1}{4}(\widehat{R}_\mu^\nu - \frac{1}{6}\delta_\mu^\nu \widehat{R}), \quad \widehat{R}_\mu^A \equiv \widehat{R}_{\mu\nu}^{AB} \delta_B^A, \quad \widehat{R} \equiv \widehat{R}_\mu^A \delta_A^\mu. \quad (\text{C.13})$$

Plugging this solution in the Lagrangian (C.10) we obtain the standard higher-derivative Weyl Lagrangian,

$$\mathcal{L} = R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2. \quad (\text{C.14})$$

We now turn to derivation of ordinary-derivative Lagrangian. To this end we introduce a decomposition of f_μ^ν into symmetric and antisymmetric tensor fields,

$$f_\nu^\mu = \frac{1}{2} s^\mu{}_\nu + \frac{1}{2} a^\mu{}_\nu, \quad (\text{C.15})$$

where

$$s^{\mu\nu} = s^{\nu\mu}, \quad a^{\mu\nu} = -a^{\nu\mu}, \quad (\text{C.16})$$

$$s^{\mu\nu} \equiv s^\mu{}_\rho \eta^{\rho\nu}, \quad a^{\mu\nu} \equiv a^\mu{}_\rho \eta^{\rho\nu}. \quad (\text{C.17})$$

In terms of fields $s^{\mu\nu}$ and $a^{\mu\nu}$, the Lagrangian (C.10) takes the form

$$\begin{aligned} \mathcal{L} = & -s^{\mu\nu}(R_{\mu\nu} + \partial_\mu b_\nu + \partial_\nu b_\mu) + \frac{1}{2}s(R + 4\partial b) \\ & - s_{\mu\nu} s^{\mu\nu} + s^2 - a^{\mu\nu} F_{\mu\nu} + a_{\mu\nu} a^{\mu\nu}, \end{aligned} \quad (\text{C.18})$$

where we use the notation

$$F_{\mu\nu} \equiv \partial_\mu b_\nu - \partial_\nu b_\mu, \quad \partial b \equiv \partial_\mu b^\mu, \quad s \equiv s_\mu^\mu. \quad (\text{C.19})$$

We now consider E.o.M for the antisymmetric tensor field $a^{\mu\nu}$,

$$\partial_{a^{\mu\nu}} \mathcal{L} = 0, \quad (\text{C.20})$$

which allows us to solve the field $a^{\mu\nu}$ in terms of the field strength $F^{\mu\nu}$,

$$a^{\mu\nu} = \frac{1}{2}F^{\mu\nu}. \quad (\text{C.21})$$

Plugging this solution for $a^{\mu\nu}$ into Lagrangian (C.18) we obtain

$$\begin{aligned} \mathcal{L} = & -s^{\mu\nu}(R_{\mu\nu} + \partial_\mu b_\nu + \partial_\nu b_\mu) + \frac{1}{2}s(R + 4\partial b) \\ & - s_{\mu\nu}s^{\mu\nu} + s^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \end{aligned} \quad (\text{C.22})$$

This is nothing but the ordinary-derivative Lagrangian²⁹. The Lagrangian obtained is invariant with respect to gauge transformations

$$\delta h^{\mu\nu} = \partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu + 2\eta^{\mu\nu} \epsilon_{(D)}, \quad (\text{C.23})$$

$$\delta s^{\mu\nu} = \partial^\mu \epsilon_{(K)}^\nu + \partial^\nu \epsilon_{(K)}^\mu, \quad (\text{C.24})$$

$$\delta b^\mu = \partial^\mu \epsilon_{(D)} - 2\epsilon_{(K)}^\mu. \quad (\text{C.25})$$

Generalization to general gravitational background. Lagrangian (C.22) can be extended to general gravitational background. Using gauge invariant Lagrangian in [38] we find the following ordinary-derivative Lagrangian in general gravitational background:

$$\begin{aligned} \frac{1}{\sqrt{-g}}\mathcal{L} = & -s^{\mu\nu}(R_{\mu\nu} + D_\mu b_\nu + D_\nu b_\mu + 2b_\mu b_\nu) + \frac{1}{2}s(R + 4Db - 2b^2) \\ & - s_{\mu\nu}s^{\mu\nu} + s^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \end{aligned} \quad (\text{C.26})$$

where $g = \det g_{\mu\nu}$ and $g_{\mu\nu}$ is a metric tensor of general gravitational background, $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$, $R = R^\mu{}_\mu$, $R^\mu{}_{\nu\rho\lambda} = \partial_\rho \Gamma^\mu_{\nu\lambda} + \dots$. In (C.26)-(C.33), μ, ν, ρ are indices of curved gravitational background. The remaining notation is

$$s \equiv g^{\mu\nu}s_{\mu\nu}, \quad Db \equiv D_\mu b^\mu, \quad b^2 \equiv g^{\mu\nu}b_\mu b_\nu, \quad (\text{C.27})$$

and D_μ stands for covariant derivative.

Lagrangian (C.26) is invariant with respect to gauge conformal boost and Weyl transformations,

$$\delta g_{\mu\nu} = 2g_{\mu\nu}\epsilon_{(D)}, \quad (\text{C.28})$$

$$\delta s_{\mu\nu} = D_\mu \epsilon_{(K)\nu} + D_\nu \epsilon_{(K)\mu} + 2(\epsilon_{(K)\mu} b_\nu + \epsilon_{(K)\nu} b_\mu - g_{\mu\nu} \epsilon_{(K)\rho} b^\rho), \quad (\text{C.29})$$

$$\delta b_\mu = \partial_\mu \epsilon_{(D)} - 2\epsilon_{(K)\mu}, \quad (\text{C.30})$$

²⁹ Fields in (C.22), are identified with the ones in (5.78) in obvious way: $h^{\mu\nu} = 2\phi_{2,-1}^{\mu\nu}$, $s^{\mu\nu} = \frac{1}{2}\phi_{2,1}^{\mu\nu}$, $b^\mu = -\phi_{1,0}^\mu$.

and the standard diffeomorphism transformations

$$\delta g_{\mu\nu} = (\xi\partial)g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}, \quad (\text{C.31})$$

$$\delta s_{\mu\nu} = (\xi\partial)s_{\mu\nu} + \partial_\mu \xi^\rho s_{\rho\nu} + \partial_\nu \xi^\rho s_{\mu\rho}, \quad (\text{C.32})$$

$$\delta b_\mu = (\xi\partial)b_\mu + \partial_\mu \xi^\rho b_\rho, \quad (\text{C.33})$$

where we use the notation $\xi\partial = \xi^\mu \partial_\mu$.

Appendix D Counting of on-shell D.o.F for spin 3/2 conformal field

We analyze on-shell D.o.F for spin 3/2 conformal field that is described by the standard higher-derivative Lagrangian given in (7.1). Lagrangian (7.1) leads to E.o.M,

$$\square^k P_{3/2}^{AB} \not{\partial} \psi^B = 0. \quad (\text{D.1})$$

We now analyze E.o.M (D.1) by using light-cone gauge to fix gauge symmetry (7.6) and gamma-transversality condition to fix gauge symmetry (7.7),

$$\psi^+ = 0, \quad (\text{D.2})$$

$$\gamma^A \psi^A = 0. \quad (\text{D.3})$$

Using gauge conditions (D.2),(D.3), one can prove that E.o.M amount to the following equations (for details see below):

$$\square^k \not{\partial} \psi^A = 0, \quad (\text{D.4})$$

$$\square^{k-1} \not{\partial} \partial \psi = 0, \quad (\text{D.5})$$

$\partial \psi \equiv \partial^A \psi^A$. Making use of the notation

$$\psi_{1,-k}^A \equiv \psi^A, \quad (\text{D.6})$$

we transform Eqs.(D.4), (D.5) into ordinary-derivative form,

$$\not{\partial} \psi_{1,k'}^A + \psi_{1,k'+1}^A = 0, \quad k' = -k, -k+1, \dots, k-1, k; \quad (\text{D.7})$$

$$\not{\partial} \psi_{0,k'} + \psi_{0,k'+1} = 0, \quad k' = -k+1, -k+2, \dots, k-2, k-1; \quad (\text{D.8})$$

$$\gamma \psi_{1,k'} = 0, \quad k' = -k, -k+2, \dots, k-2, k; \quad (\text{D.9})$$

$$\gamma \psi_{1,k'} + \psi_{0,k'} = 0, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (\text{D.10})$$

where we use the conventions $\gamma\psi_{1,k'} \equiv \gamma^A\psi_{1,k'}^A$,

$$\psi_{1,k+1}^A \equiv 0, \quad \psi_{0,-k} \equiv 0, \quad \psi_{0,k} \equiv 0. \quad (\text{D.11})$$

Simple way to derive (D.7)-(D.10) is to note the representation for the fields $\psi_{1,k'}^A, \psi_{0,k'}$ in terms of the generic field $\psi_{1,-k}^A$:

$$\psi_{1,k'}^A = \square^{\frac{k+k'}{2}} \psi_{1,-k}^A, \quad k' = -k, -k+2, \dots, k-2, k; \quad (\text{D.12})$$

$$\psi_{1,k'}^A = -\square^{\frac{k-1+k'}{2}} \not\partial \psi_{1,-k}^A, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (\text{D.13})$$

$$\psi_{0,k'} = 2\square^{\frac{k-1+k'}{2}} \partial \psi_{1,-k}, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (\text{D.14})$$

$$\psi_{0,k'} = -2\square^{\frac{k-2+k'}{2}} \not\partial \partial \psi_{1,-k}, \quad k' = -k+2, -k+4, \dots, k-4, k-2. \quad (\text{D.15})$$

We note that light-cone gauge (D.2) implies

$$\psi_{1,k'}^+ = 0, \quad k' = -k, -k+1, \dots, k-1, k. \quad (\text{D.16})$$

Before to proceed, we introduce the notation

$$\psi_{1,k'}^{A\hat{+}} \equiv \Pi^{\hat{+}} \psi_{1,k'}^A, \quad \psi_{1,k'}^{A\hat{-}} \equiv \Pi^{\hat{-}} \psi_{1,k'}^A, \quad \psi_{0,k'}^{\hat{+}} \equiv \Pi^{\hat{+}} \psi_{0,k'}, \quad \psi_{0,k'}^{\hat{-}} \equiv \Pi^{\hat{-}} \psi_{0,k'}, \quad (\text{D.17})$$

$$\Pi^{\hat{+}} \equiv \frac{1}{2} \gamma^- \gamma^+, \quad \Pi^{\hat{-}} \equiv \frac{1}{2} \gamma^+ \gamma^-. \quad (\text{D.18})$$

Some of the fields in (D.17), namely

$$\psi_{1,k'}^{I\hat{+}}, \quad k' = -k, -k+1, \dots, k-1, k; \quad (\text{D.19})$$

$$\psi_{0,k'}^{\hat{+}}, \quad k' = -k+1, -k+2, \dots, k-2, k-1, \quad (\text{D.20})$$

turn out to be dynamical on-shell D.o.F. This to say that using E.o.M (D.7),(D.8), algebraic constraints (D.9),(D.10), and light-cone gauge (D.16) we can solve the remaining fields in (D.17) in terms of on-shell dynamical D.o.F. (D.19),(D.20),

$$\psi_{1,k'}^{I\hat{-}} = -\frac{\gamma^+}{2\partial^+} \gamma^J \partial^J \psi_{1,k'}^{I\hat{+}} - \frac{\gamma^+}{2\partial^+} \psi_{1,k'+1}^{I\hat{+}}, \quad (\text{D.21})$$

$$\psi_{1,k'}^{-\hat{+}} = -\frac{\partial^J}{\partial^+} \psi_{1,k'}^{J\hat{+}} + \frac{1}{2\partial^+} \psi_{0,k'+1}^{\hat{+}}, \quad (\text{D.22})$$

$$\psi_{1,k'}^{-\hat{-}} = -\frac{\gamma^+}{2\partial^+} \gamma^J \partial^J \psi_{1,k'}^{-\hat{+}} - \frac{\gamma^+}{2\partial^+} \psi_{1,k'+1}^{-\hat{+}}, \quad (\text{D.23})$$

$$\psi_{0,k'}^{\hat{-}} = -\frac{\gamma^+}{2\partial^+} \gamma^J \partial^J \psi_{0,k'}^{\hat{+}} - \frac{\gamma^+}{2\partial^+} \psi_{0,k'+1}^{\hat{+}}. \quad (\text{D.24})$$

Relations (D.21), (D.22), and (D.24) are obtained from (D.7), (D.8), while relation (D.23) is obtained from (D.21) and the following relations:

$$\psi_{1,k'}^{-\hat{+}} = -\frac{1}{2}\gamma^{-}\gamma^I\psi_{1,k'}^{I\hat{-}}, \quad k' = -k, -k+2, \dots, k-2, k; \quad (\text{D.25})$$

$$\gamma^I\psi_{1,k'}^{I\hat{+}} = 0, \quad k' = -k, -k+2, \dots, k-2, k; \quad (\text{D.26})$$

$$\psi_{1,k'}^{-\hat{+}} = -\frac{1}{2}\gamma^{-}\gamma^I\psi_{1,k'}^{I\hat{-}} - \frac{1}{2}\gamma^{-}\psi_{0,k'}^{\hat{-}}, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (\text{D.27})$$

$$\gamma^I\psi_{1,k'}^{I\hat{+}} + \psi_{0,k'}^{\hat{+}} = 0, \quad k' = -k+1, -k+3, \dots, k-3, k-1; \quad (\text{D.28})$$

which are obtainable from algebraic constraints (D.9),(D.10).

Thus, we obtained on-shell D.o.F (D.19), (D.20) which satisfy the algebraic constraints (D.26), (D.28) and the constraints $\Pi^{\hat{-}}\psi_{0,k'}^{I\hat{+}} = 0$, $\Pi^{\hat{-}}\psi_{0,k'}^{\hat{+}} = 0$. It is easy to see that such on-shell D.o.F are in one-to-one correspondence with on-shell D.o.F (7.8)-(7.11) subject to the constraints $\gamma^I\psi_{0,k'}^I = 0$, $\Pi^{\hat{-}}\psi_{0,k'}^I = 0$, $\Pi^{\hat{-}}\psi_{0,k'} = 0$.

We finish with details of derivation of Eqs.(D.4), (D.5). Using gauge conditions (D.2),(D.3) we obtain relation

$$\square^k P_{3/2}^{+B}\psi^B = -\frac{\square^{k-1}}{d-1}\left((d-2)\partial^+\not{\partial}\partial\psi + \gamma^+\square\partial\psi\right), \quad (\text{D.29})$$

which, together with (D.1), leads to equation

$$\square^{k-1}\left((d-2)\partial^+\not{\partial}\partial\psi + \gamma^+\square\partial\psi\right) = 0. \quad (\text{D.30})$$

Multiplying this equation by γ^+ gives

$$\square^{k-1}\gamma^+\partial^+\not{\partial}\partial\psi = 0. \quad (\text{D.31})$$

Eq.(D.31) implies relation

$$\square^{k-1}\gamma^+\square\partial\psi = \square^{k-1}\gamma^+\not{\partial}\not{\partial}\partial\psi = \square^{k-1}2\partial^+\not{\partial}\partial\psi. \quad (\text{D.32})$$

Plugging (D.32) in (D.30) and taking into account assumption that kernel of the operator ∂^+ is trivial we obtain the desired Eq.(D.5). Finally, plugging the gauge condition (D.3) and Eq.(D.5) in E.o.M (D.1) we obtain Eq.(D.4).

References

- [1] E. S. Fradkin and A. A. Tseytlin, Phys. Rept. **119**, 233 (1985).
- [2] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].
- [3] S.E.Konstein, M.A.Vasiliev and V.N.Zaikin, JHEP **0012**, 018 (2000) arXiv:hep-th/0010239
- [4] J. Erdmenger, Class. Quant. Grav. **14**, 2061 (1997) [arXiv:hep-th/9704108].
- [5] A. Y. Segal, Nucl. Phys. B **664**, 59 (2003) [arXiv:hep-th/0207212].
- [6] O. V. Shaynkman, I. Y. Tipunin and M. A. Vasiliev, Rev. Math. Phys. **18**, 823 (2006) hep-th/0401086
- [7] V. Balasubramanian, P. Kraus and A. E. Lawrence, Phys. Rev. D **59**, 046003 (1999) hep-th/9805171
- [8] V. K. Dobrev, Nucl. Phys. B **553**, 559 (1999) [arXiv:hep-th/9812194].
- [9] R. R. Metsaev, Nucl. Phys. B **563**, 295 (1999) [arXiv:hep-th/9906217].
- [10] V. Balasubramanian, E. G. Gimon, D. Minic and J. Rahmfeld, Phys. Rev. D **63**, 104009 (2001) [arXiv:hep-th/0007211].
- [11] R. R. Metsaev, Phys. Lett. B **636**, 227 (2006) [arXiv:hep-th/0512330].
- [12] A. Petkou, Annals Phys. **249**, 180 (1996) [arXiv:hep-th/9410093].
- [13] A. C. Petkou, Phys.Lett. B **389**, 18 (1996) [arXiv:hep-th/9602054].
- [14] H. Liu and A. A. Tseytlin, Nucl. Phys. B **533**, 88 (1998) [arXiv:hep-th/9804083].
- [15] J. Isberg, U. Lindstrom and B. Sundborg, Phys. Lett. B **293**, 321 (1992) [arXiv:hep-th/9207005].
- [16] G. Bonelli, Nucl. Phys. B **669**, 159 (2003) [arXiv:hep-th/0305155].
- [17] G. Bonelli, JHEP **0311**, 028 (2003) [arXiv:hep-th/0309222].
- [18] Yu. M. Zinoviev, arXiv:hep-th/0108192.
- [19] R. R. Metsaev, Phys. Lett. B **643**, 205 (2006) [arXiv:hep-th/0609029].
- [20] M. A. Vasiliev, Phys. Lett. B **243**, 378 (1990).
- [21] M. A. Vasiliev, arXiv:0707.1085 [hep-th].
- [22] V. E. Lopatin and M. A. Vasiliev, Mod. Phys. Lett. A **3**, 257 (1988).
- [23] M. A. Vasiliev, Nucl. Phys. B **301**, 26 (1988).
- [24] J. M. F. Labastida, Nucl. Phys. B **322**, 185 (1989).
- [25] K. Hallowell and A. Waldron, Nucl. Phys. B **724**, 453 (2005) [arXiv:hep-th/0505255].
- [26] T. Biswas and W. Siegel, JHEP **0207**, 005 (2002) [arXiv:hep-th/0203115].
- [27] N. T. Evans, J. Math. Phys. **8**, 170 (1967).

- [28] G. Mack, Commun. Math. Phys. **55**, 1 (1977).
- [29] R. R. Metsaev, Phys. Lett. B **354**, 78 (1995).
- [30] W. Siegel, Int. J. Mod. Phys. A **4**, 2015 (1989).
- [31] R. R. Metsaev, Mod. Phys. Lett. A **10**, 1719 (1995).
- [32] V. K. Dobrev and V. B. Petkova, Phys. Lett. B **162**, 127 (1985).
- [33] M. Gunaydin, D. Minic and M. Zagermann, Nucl. Phys. B **534**, 96 (1998) hep-th/9806042
- [34] I. L. Buchbinder and S. L. Lyakhovich, Class. Quant. Grav. **4** (1987) 1487.
- [35] F. J. de Urries, J. Julve and E. J. Sanchez, J. Phys. A **34**, 8919 (2001) [arXiv:hep-th/0105301].
- [36] E. J. S. Villasenor, J. Phys. A **35**, 6169 (2002) [arXiv:hep-th/0203197].
- [37] S. C. Lee and P. van Nieuwenhuizen, Phys. Rev. D **26**, 934 (1982).
- [38] M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, Phys. Lett. B **69**, 304 (1977).
- [39] E. Bergshoeff, M. de Roo and B. de Wit, Nucl. Phys. B **182**, 173 (1981).
- [40] E. Bergshoeff, M. de Roo and B. de Wit, Nucl. Phys. B **217**, 489 (1983).
- [41] E. Bergshoeff, S. Cucu, M. Derix, T. de Wit, R. Halbersma and A. Van Proeyen, JHEP **0106**, 051 (2001) [arXiv:hep-th/0104113].
- [42] E. Bergshoeff, E. Sezgin and A. Van Proeyen, Class. Quant. Grav. **16**, 3193 (1999) hep-th/9904085
- [43] E. Bergshoeff, A. Salam and E. Sezgin, Nucl. Phys. B **279**, 659 (1987).
- [44] E. A. Bergshoeff and M. de Roo, Phys. Lett. B **138**, 67 (1984).
- [45] N. Boulanger and M. Henneaux, Annalen Phys. **10**, 935 (2001) [arXiv:hep-th/0106065].
- [46] E. S. Fradkin and V. Y. Linetsky, Phys. Lett. B **231**, 97 (1989).
- [47] E. S. Fradkin and V. Y. Linetsky, Nucl. Phys. B **350**, 274 (1991).
- [48] Yu. M. Zinoviev, Nucl. Phys. B **770**, 83 (2007) [arXiv:hep-th/0609170].
- [49] R. R. Metsaev, arXiv:hep-th/0612279.
- [50] Yu. M. Zinoviev, JHEP **0705**, 092 (2007) [arXiv:hep-th/0703118].
- [51] Yu. M. Zinoviev, arXiv:0704.1535 [hep-th].
- [52] W. Siegel, arXiv:hep-th/9912205.
- [53] K. S. Nirov, Int. J. Mod. Phys. A **11**, 5279 (1996) [arXiv:hep-th/9412134].
- [54] I. L. Buchbinder, A. Pashnev and M. Tsulaia, Phys. Lett. B **523**, 338 (2001) [arXiv:hep-th/0109067].
- [55] X. Bekaert, I. L. Buchbinder, A. Pashnev and M. Tsulaia, Class. Quant. Grav. **21**, S1457 (2004) [arXiv:hep-th/0312252].
- [56] I. L. Buchbinder, V. A. Krykhtin and A. Pashnev, Nucl. Phys. B **711**, 367 (2005) hep-th/0410215

- [57] I. L. Buchbinder and V. A. Krykhtin, Nucl. Phys. B **727**, 537 (2005) [arXiv:hep-th/0505092].
- [58] I. L. Buchbinder, V. A. Krykhtin and P. M. Lavrov, Nucl. Phys. B **762**, 344 (2007) hep-th/0608005
- [59] I. L. Buchbinder, V. A. Krykhtin and A. A. Reshetnyak, arXiv:hep-th/0703049.
- [60] D. Francia and A. Sagnotti, Phys. Lett. B **543**, 303 (2002) [arXiv:hep-th/0207002].
- [61] A. Sagnotti and M. Tsulaia, Nucl. Phys. B **682**, 83 (2004) [arXiv:hep-th/0311257].
- [62] D. Francia and A. Sagnotti, Phys. Lett. B **624**, 93 (2005) [arXiv:hep-th/0507144].
- [63] C. S. Aulakh, I. G. Koh and S. Ouvry, Phys. Lett. B **173**, 284 (1986).
- [64] J. M. F. Labastida and T. R. Morris, Phys. Lett. B **180**, 101 (1986).
- [65] C. Burdik, A. Pashnev and M. Tsulaia, Mod. Phys. Lett. A **16**, 731 (2001) [arXiv:hep-th/0101201].
- [66] X. Bekaert and N. Boulanger, Commun. Math. Phys. **245**, 27 (2004) [arXiv:hep-th/0208058].
- [67] Yu. M. Zinoviev, [arXiv:hep-th/0306292].
- [68] K. B. Alkalaev, O. V. Shaynkman and M. A. Vasiliev, arXiv:hep-th/0601225.
- [69] K. B. Alkalaev, [arXiv:hep-th/0501105].
- [70] P. Y. Moshin and A. A. Reshetnyak, arXiv:0707.0386 [hep-th].
- [71] L. Brink, R. R. Metsaev and M. A. Vasiliev, Nucl. Phys. B **586**, 183 (2000) [arXiv:hep-th/0005136].
- [72] R. R. Metsaev and A. A. Tseytlin, Phys. Rev. D **63**, 046002 (2001) [arXiv:hep-th/0007036].
- [73] R. R. Metsaev, C. B. Thorn and A. A. Tseytlin, Nucl. Phys. B **596**, 151 (2001) [arXiv:hep-th/0009171].
- [74] R. R. Metsaev, Nucl. Phys. B **759**, 147 (2006) [arXiv:hep-th/0512342].